P3. Quantum Treatment: One-Photon System

- Consider a single photon which is in a momentum eigenstate $\vec{q}$. The quantum mechanical state space of the photon has two remaining degrees of freedom, representing photon spin, or polarization. We can choose as the basis vectors of this state space $|x\rangle$ and $|y\rangle$, which represent linear polarization in the x and y directions.

An arbitrary photon polarization state vector is then

$$\underline{|\varepsilon\rangle = a_x e^{i\alpha_x}|x\rangle + a_y e^{i\alpha_y}|y\rangle} \qquad (1)$$

Here $|x\rangle$ and $|y\rangle$ are orthonormal: $\langle x|x\rangle = \langle y|y\rangle = 1$, $\langle x|y\rangle = \langle y|x\rangle^* = 0$.

Normalization of state

$$\langle\varepsilon|\varepsilon\rangle = \left(a_x e^{-i\alpha_x}\langle x| + a_y e^{-i\alpha_y}\langle y|\right)\left(a_x e^{i\alpha_x}|x\rangle + a_y e^{i\alpha_y}|y\rangle\right) = a_x^2 + a_y^2 = 1.$$

The probability of measuring the photon to have linear x (or y) polarization is a_x^2 (or a_y^2).

- The one-photon quantum-mechanical operators for the Stokes parameters are

$$\boxed{\begin{aligned} \hat{I} &\equiv |x\rangle\langle x| + |y\rangle\langle y| \\ \hat{Q} &\equiv |x\rangle\langle x| - |y\rangle\langle y| \\ \hat{U} &\equiv |x\rangle\langle y| + |y\rangle\langle x| \\ \hat{V} &\equiv i|y\rangle\langle x| - i|x\rangle\langle y| \end{aligned}} \qquad (2)$$

- Using

$$\langle x|\varepsilon\rangle = a_x e^{i\alpha_x} \qquad \langle\varepsilon|x\rangle = a_x e^{-i\alpha_x}$$
$$\langle y|\varepsilon\rangle = a_y e^{i\alpha_y} \qquad \langle\varepsilon|y\rangle = a_y e^{-i\alpha_y}$$

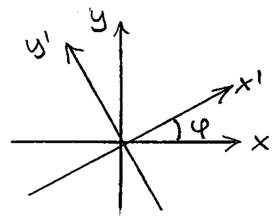
we find the expectation values for the Stokes parameters

$$\boxed{\begin{aligned} I &\equiv \langle\hat{I}\rangle = \langle\varepsilon|\hat{I}|\varepsilon\rangle = \langle\varepsilon|x\rangle\langle x|\varepsilon\rangle + \langle\varepsilon|y\rangle\langle y|\varepsilon\rangle = a_x^2 + a_y^2 = 1 \\ Q &\equiv \langle\hat{Q}\rangle = \langle\varepsilon|\hat{Q}|\varepsilon\rangle = \langle\varepsilon|x\rangle\langle x|\varepsilon\rangle - \langle\varepsilon|y\rangle\langle y|\varepsilon\rangle = a_x^2 - a_y^2 \\ U &\equiv \langle\hat{U}\rangle = \langle\varepsilon|\hat{U}|\varepsilon\rangle = \langle\varepsilon|x\rangle\langle y|\varepsilon\rangle + \langle\varepsilon|y\rangle\langle x|\varepsilon\rangle = 2a_x a_y \cos(\alpha_y - \alpha_x) \\ V &\equiv \langle\hat{V}\rangle = \langle\varepsilon|\hat{V}|\varepsilon\rangle = i\langle\varepsilon|y\rangle\langle x|\varepsilon\rangle - i\langle\varepsilon|x\rangle\langle y|\varepsilon\rangle = 2a_x a_y \sin(\alpha_y - \alpha_x) \end{aligned}} \qquad (3)$$

$$\therefore \quad \underline{Q^2 + U^2 + V^2 = I^2 = 1}$$

Thus a single photon is always "fully polarized".

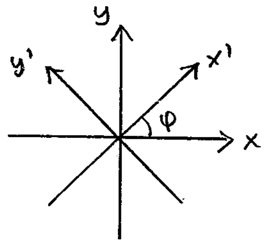
Rotation of Coordinate System



- If we rotate the xy coord. axes and define new basis state vectors $|x'\rangle$ and $|y'\rangle$ their relation to the old ones is

$$\boxed{\begin{aligned} |x'\rangle &= \cos\varphi\,|x\rangle + \sin\varphi\,|y\rangle \\ |y'\rangle &= -\sin\varphi\,|x\rangle + \cos\varphi\,|y\rangle \end{aligned}} \quad \Rightarrow \quad \begin{aligned} |x\rangle &= \cos\varphi\,|x'\rangle - \sin\varphi\,|y'\rangle \\ |y\rangle &= \sin\varphi\,|x'\rangle + \cos\varphi\,|y'\rangle \end{aligned}$$

- In particular, if the rotation is by 45°,



$$|x'\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + |y\rangle\right) \qquad |x\rangle = \frac{1}{\sqrt{2}}\left(|x'\rangle - |y'\rangle\right)$$

$$|y'\rangle = \frac{1}{\sqrt{2}}\left(-|x\rangle + |y\rangle\right) \qquad |y\rangle = \frac{1}{\sqrt{2}}\left(|x'\rangle + |y'\rangle\right)$$

which illustrates the meaning of the $\hat{U}$ operator:

$$\hat{U} = |x\rangle\langle y| + |y\rangle\langle x| = \frac{1}{2}\left(|x'\rangle - |y'\rangle\right)\left(\langle x'| + \langle y'|\right) + \frac{1}{2}\left(|x'\rangle + |y'\rangle\right)\left(\langle x'| - \langle y'|\right)$$

$$= \frac{1}{2}\left(|x'\rangle\langle x'| + \cancel{|x'\rangle\langle y'|} - \cancel{|y'\rangle\langle x'|} - |y'\rangle\langle y'| + |x'\rangle\langle x'| - \cancel{|x'\rangle\langle y'|} + \cancel{|y'\rangle\langle x'|} - |y'\rangle\langle y'|\right)$$

$$= |x'\rangle\langle x'| - |y'\rangle\langle y'| = \hat{Q}'$$

Thus $U = \langle \hat{U} \rangle = a_{x'}^2 - a_{y'}^2$ is the difference between the two probabilities of observing the photon in the 45° and 135° (= −45°) linear polarization states; just like $Q = \langle \hat{Q} \rangle = a_x^2 - a_y^2$ is the difference between the probabilities of observing 0° and 90° linear polarization states.

Helicity Basis

- We can also choose as the basis of the photon polarization state space

$$\boxed{\begin{aligned} |+\rangle &\equiv \tfrac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) \\ |-\rangle &\equiv \tfrac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right) \end{aligned}} \quad \Rightarrow \quad \begin{aligned} |x\rangle &= \tfrac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \\ |y\rangle &= \tfrac{-i}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \end{aligned}$$

The helicity basis $\{|+\rangle, |-\rangle\}$ corresponds to the two opposite circular polarization states, or the two photon spin states $s_z = \pm 1$.

- In the helicity basis

$$\underline{\hat{I}} = |x\rangle\langle x| + |y\rangle\langle y| = \tfrac{1}{2}\left(|+\rangle + |-\rangle\right)\left(\langle +| + \langle -|\right) + \tfrac{1}{2}\left(|+\rangle - |-\rangle\right)\left(\langle +| - \langle -|\right) = \dots = \underline{|+\rangle\langle +| + |-\rangle\langle -|}$$

$\hat{Q} = \dots$ (exercise)

$\hat{U} = \dots$

$$\underline{\hat{V}} = i|y\rangle\langle x| - i|x\rangle\langle y| = \tfrac{1}{2}\left(|+\rangle - |-\rangle\right)\left(\langle +| + \langle -|\right) + \tfrac{1}{2}\left(|+\rangle + |-\rangle\right)\left(\langle +| - \langle -|\right)$$

$$= \tfrac{1}{2}\left(|+\rangle\langle +| + \cancel{|+\rangle\langle -|} - \cancel{|-\rangle\langle +|} - |-\rangle\langle -| + |+\rangle\langle +| - \cancel{|+\rangle\langle -|} + \cancel{|-\rangle\langle +|} - |-\rangle\langle -|\right)$$

$$= \underline{|+\rangle\langle +| - |-\rangle\langle -|}$$

- An arbitrary state vector can be written $\underline{|\varepsilon\rangle = a_+ e^{i\alpha_+}|+\rangle + a_- e^{i\alpha_-}|-\rangle}$

Thus $I \equiv \langle \hat{I} \rangle = a_+^2 + a_-^2 = 1$

and $\underline{V \equiv \langle \hat{V} \rangle = a_+^2 - a_-^2}$ is the difference between the probabilities at the positive and negative helicity (circular polarization) states, or $S_z = +1$ and $S_z = -1$ (where $\hat{z}$ points in the direction of photon momentum).

- Rotating the xy-crd. axes by an angle φ defines a new helicity basis

$$\underline{|+'\rangle} = \tfrac{1}{\sqrt{2}}\left(|x'\rangle + i|y'\rangle\right) = \dots = \cos\varphi \cdot \tfrac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) - i\sin\varphi \cdot \tfrac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) = \underline{e^{-i\varphi}|+\rangle}$$

$$\underline{|-'\rangle} = \tfrac{1}{\sqrt{2}}\left(|x'\rangle - i|y'\rangle\right) = \dots = \cos\varphi \cdot \tfrac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right) + i\sin\varphi \cdot \tfrac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right) = \underline{e^{+i\varphi}|-\rangle}$$

$$\Rightarrow \quad \begin{aligned} |+\rangle &= e^{+i\varphi}|+'\rangle \\ |-\rangle &= e^{-i\varphi}|-'\rangle \end{aligned} \qquad \text{and} \qquad \begin{aligned} |\varepsilon\rangle &= a_+ e^{i\alpha_+}|+\rangle + a_- e^{i\alpha_-}|-\rangle \\ &= a_+ e^{i\alpha_+ + i\varphi}|+'\rangle + a_- e^{i\alpha_- - i\varphi}|-'\rangle \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} a_+' &= a_+ \qquad & \alpha_+' &= \alpha_+ + \varphi \\ a_-' &= a_- \qquad & \alpha_-' &= \alpha_- - \varphi \end{aligned}$$

$\therefore$ For an arbitrary polarization state, the probability of observing + (or −) polarization, a_+^2 (or a_-^2) is independent of orientation of the xy crd. system (whereas the probabilities at observing x (or y) polarization of course depends on it).