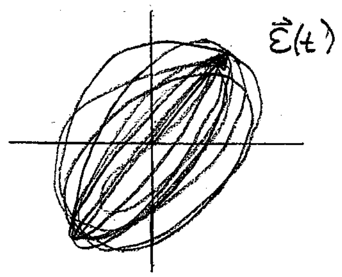
P2. Classical Quasi-Monochromatic Plane Wave

- We saw that, classically, a monochromatic plane wave is necessarily fully polarized ($I^2 = Q^2 + U^2 + V^2$). However, in nature we encounter unpolarized and partially polarized radiation. While this can be understood through the quantum nature of radiation; we can also note that the monochromatic (a single well-defined frequency) wave is an idealization. For example, CMB has a blackbody spectrum, and CMB detectors are sensitive to a frequency band $\Delta\nu$, instead of of a delta-function response $\delta(\nu - \nu_0)$.

  We can obtain partially polarized or unpolarized radiation as a superposition of polarized waves

Figure. Superposition of (a) linearly polarized, (b) circularly polarized plane waves

  Classically, the superposition requires adding together waves with different frequencies. Instead of an ellipse, the electric field vector $\vec{E}(t)$ draws then a more complicated pattern, and to determine the polarization state, we need to extend the measurement over many oscillation patterns.



- Consider thus a plane wave with a well-defined direction $\hat{n}$, but consisting of a superposition of frequencies in a narrow range $\Delta\omega$ around a central frequency ω. The electric field (at, say, $\vec{x} = 0$) can then be given as

$$\underline{\vec{E}(t) = \mathrm{Re}\left[(E_x \hat{e}_x + E_y \hat{e}_y)\, e^{-i\omega t} \right]} \qquad (1)$$

where the complex amplitude $\vec{E} = \vec{E}(t)$,

$$\boxed{\begin{aligned} E_x &= E_x(t) = a_x(t)\, e^{i\alpha_x(t)} \\ E_y &= E_y(t) = a_y(t)\, e^{i\alpha_y(t)} \end{aligned}} \qquad (2)$$

varies slowly in time. (Slowly compared to the oscillation frequency ω, which for CMB is many GHz; but fast compared to the measurement frequency, which for CMB anisotropy measurements (e.g., Planck) is many Hz.)

- We now define Stokes parameters as time averages (denoted by $\langle\cdot\rangle$)

$$\boxed{\begin{aligned} I &\equiv \langle |E_x|^2\rangle + \langle |E_y|^2\rangle &&= \langle a_x^2\rangle + \langle a_y^2\rangle \\ Q &\equiv \langle |E_x|^2\rangle - \langle |E_y|^2\rangle &&= \langle a_x^2\rangle - \langle a_y^2\rangle \\ U &\equiv \langle E_x E_y^*\rangle + \langle E_y E_x^*\rangle &&= 2\,\mathrm{Re}\langle E_x E_y^*\rangle = 2\langle a_x a_y \cos\alpha\rangle \\ V &\equiv i\langle E_x E_y^*\rangle - i\langle E_y E_x^*\rangle &&= 2\,\mathrm{Im}\langle E_x^* E_y\rangle = 2\langle a_x a_y \sin\alpha\rangle \end{aligned}} \qquad (3)$$

where $\underline{\alpha(t) \equiv \alpha_y(t) - \alpha_x(t)}$. (4)

- It can be shown (exercise), that now $\underline{0 \le Q^2+U^2+V^2 \le I^2}$ (5)

and completely unpolarized radiation $(Q=U=V=0)$ is now possible.

- The transformation properties under xy-rotation are still given by Eq. (1.13). (The time averaging does not interfere w the old transformation.)

- We can also define the coherence tensor

$$\boxed{\begin{aligned} C_{ij} &\equiv \langle E_i E_j^*\rangle = \langle a_i a_j e^{i(\alpha_i-\alpha_j)}\rangle \qquad (6) \\ &= \begin{bmatrix} \langle a_x^2\rangle & \langle a_x a_y e^{-i\alpha}\rangle \\ \langle a_y a_x e^{i\alpha}\rangle & \langle a_y^2\rangle \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^*\rangle & \langle E_x E_y^*\rangle \\ \langle E_y E_x^*\rangle & \langle E_y E_y^*\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix} \end{aligned}}$$

which is Hermitian, $C_{ij}^* = C_{ji}$, with $I = \mathrm{Tr}\, C$. It can be written in terms of the Pauli spin matrices

$$\underline{\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}} \qquad (7)$$

as $\underline{C = \frac{1}{2}(I\mathbb{1} + Q\sigma_3 + U\sigma_1 + V\sigma_2)}$ (8) $\qquad \mathbb{1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

- Instead of I and Q it is sometimes more convenient to use $\underline{I_x \equiv \frac{1}{2}(I+Q)}$ and $\underline{I_y \equiv \frac{1}{2}(I-Q)}$ (9)

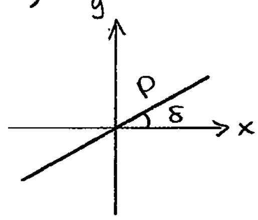
- Since $C = \langle \vec{E} \otimes \vec{E}^*\rangle$ (where $\otimes$ represents a tensor product), it transforms as a 2nd order tensor

$$C_{i'j'} \equiv \frac{1}{2}\begin{bmatrix} I'+Q' & U'-iV' \\ U'+iV' & I'-Q' \end{bmatrix} = R_{i'k} R_{j'l} C_{kl} \equiv \frac{1}{2}\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

which reproduces Eq. (1.13). (10).

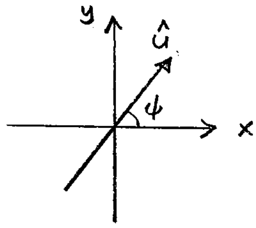
- The time evolution of the electric field $\vec{E}(t)$ no longer draws an ellipse, but we can still define

$$\boxed{\begin{aligned} p &\equiv \sqrt{Q^2+U^2} \\ \delta &\equiv \tfrac{1}{2}\arctan\frac{U}{Q} \end{aligned}} \qquad (11)$$



to describe the magnitude and direction of (the) linear (part of the) polarization, and the discussion of the linear polarization "vectors" $\tilde{p}$ and $\bar{p}$ on p. P1.5 applies.

Measurement by a linear polarimeter



- An ideal linear polarimeter is sensitive to the component of the electric field parallel to a particular direction

$$\hat{u} = \cos\psi\cdot\hat{e}_x + \sin\psi\cdot\hat{e}_y$$

i.e., to $\vec{E}(t)\cdot\hat{u}$. The measurement is proportional to $\langle(\vec{E}\cdot\hat{u})^2\rangle$.

$$\vec{E}\cdot\hat{u} = \mathrm{Re}\left[E_x e^{-i\omega t}\right]\underbrace{\hat{e}_x\cdot\hat{u}}_{\cos\psi} + \mathrm{Re}\left[E_y e^{-i\omega t}\right]\underbrace{\hat{e}_y\cdot\hat{u}}_{\sin\psi} = \mathrm{Re}\left[(E_x\cos\psi + E_y\sin\psi)e^{-i\omega t}\right]$$

$$= \frac{1}{2}\left[(E_x\cos\psi + E_y\sin\psi)e^{-i\omega t} + (E_x^*\cos\psi + E_y^*\sin\psi)e^{+i\omega t}\right]$$

$$(\vec{E}\cdot\hat{u})^2 = \frac{1}{4}\Big[\underbrace{2(E_x\cos\psi+E_y\sin\psi)(E_x^*\cos\psi+E_y^*\sin\psi)}_{\text{slowly varying piece}} + \underbrace{(E_x\cos\psi+E_y\sin\psi)^2 e^{-2i\omega t} + (E_x^*\cos\psi+E_y^*\sin\psi)^2 e^{+2i\omega t}}_{\text{rapidly varying piece}}\Big]$$

The time average of the rapidly varying piece we can approximate by zero $\Rightarrow$ the measurement is given by the slowly varying piece

$$\underline{\langle(\vec{E}\cdot\hat{u})^2\rangle} \simeq \frac{1}{2}\Big[\underbrace{\langle E_xE_x^*\rangle}_{\frac{1}{2}(I+Q)}\cos^2\psi + \underbrace{\langle E_yE_y^*\rangle}_{\frac{1}{2}(I-Q)}\sin^2\psi + \underbrace{\langle E_xE_y^* + E_yE_x^*\rangle}_{U}\sin\psi\cos\psi\Big]$$

$$= \frac{1}{4}\left[I(\cos^2\psi+\sin^2\psi) + Q(\cos^2\psi-\sin^2\psi) + U\cdot 2\sin\psi\cos\psi\right] = \underline{\frac{1}{4}(I + Q\cos 2\psi + U\sin 2\psi)}$$

- In terms of the coherence tensor we can write this result as

$$\underline{\frac{1}{2}\hat{u}\cdot C\cdot\hat{u}} = \frac{1}{2}\cdot(\cos\psi, \sin\psi)\cdot\frac{1}{2}\begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix}\begin{pmatrix}\cos\psi \\ \sin\psi\end{pmatrix} = \underline{\frac{1}{4}(I+Q\cos 2\psi + U\sin 2\psi)}$$