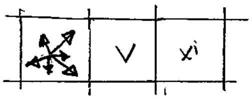
# P10. CMB Perturbations

- For the discussion of the cosmological evolution of the photon distribution, we need to take account how the photon distribution varies from place to place in the inhomogeneous universe. Earlier we used the momentum description, where a photon state was $|\vec{q}s\rangle$. Now we can imagine the universe to be divided into reference boxes (of volume V) that are small compared to cosmological scales but large compared to microscopic or instrument scales; and photon states localized in these boxes. The boxes are labeled with their space vel's $x^i$ and the 1-photon states are $|x^i \vec{q} s\rangle$.




- Thus the reduced density operator has matrix elements $\langle x^{i\prime} \vec{q}' s' | \hat{\rho}^{(1)} | x^i \vec{q} s\rangle$ but we only need the

$$\langle x^i \vec{q} s' | \hat{\rho}^{(1)} | x^i \vec{q} s\rangle \equiv \rho^{(1)}_{s's}(x^i, \vec{q}) = \langle a^\dagger_s(x^i,\vec{q})\, a_{s'}(x^i,\vec{q})\rangle \tag{1}$$

matrix elements to describe polarization and to specify the Stokes parameters

$$I(x^i,\vec{q}),\quad Q(x^i,\vec{q}),\quad U(x^i,\vec{q}),\quad V(x^i,\vec{q})$$

- As the universe evolves in time $\eta$, so does the photon reduced density matrix

$$\rho^{(1)}_{ss'}(\eta, x^i, \vec{q}) = \frac{1}{2}\begin{bmatrix} I(\eta,x^i,\vec{q}) + Q(\eta,x^i,\vec{q}) & U(\eta,x^i,\vec{q}) - iV(\eta,x^i,\vec{q}) \\ U(\eta,x^i,\vec{q}) + iV(\eta,x^i,\vec{q}) & I(\eta,x^i,\vec{q}) - Q(\eta,x^i,\vec{q}) \end{bmatrix} \tag{2}$$

- In the background universe, the photons are in thermal equilibrium (to a sufficiently good approximation for CMB anisotropy/polarization work) (*

$$\rho^{(1)}_{ss'}(\eta, x^i, \vec{q}) = \bar{\rho}_{ss'}(\eta, q) \equiv \frac{\delta_{ss'}}{e^{q/T(\eta)}-1} = \frac{1}{e^{q/T(\eta)}-1}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \tag{3}$$

so that
$$\bar{I}(\eta,q) = \frac{2}{e^{q/T(\eta)}-1}, \qquad \bar{U} = \bar{Q} = \bar{V} = 0 \tag{4}$$

- We can write the photon reduced density matrix for the perturbed universe, Eq. (2), as background + perturbation

$$\rho^{(1)}_{ss'}(\eta, x^i, \vec{q}) = \bar{\rho}_{ss'}(\eta, q) + \delta\rho^{(1)}_{ss'}(\eta, x^i, \vec{q}) \tag{5}$$

where
$$\delta\rho^{(1)}_{ss'}(\eta, x^i, \vec{q}) = \frac{1}{2}\begin{bmatrix} \delta I + Q & U - iV \\ U + iV & \delta I - Q \end{bmatrix} \tag{6}$$

*) We now revert to natural units, where $c = \hbar = k_B = \varepsilon_0 = \mu_0 = 1$.

- We now follow the discussion at §F4 (The Brightness Equation). We note that the photon distribution function $f(\eta, x^i; \vec{q})$ at §F4 is essentially the same as our Stokes parameter $I(\eta, x^i; \vec{q})$,

$$f(\eta, x^i; \vec{q}) = \frac{1}{h^3} I(\eta, x^i; \vec{q}) = \frac{1}{(2\pi)^3} I(\eta, x^i; \vec{q}) \tag{7}$$

The phase space density of location & momentum states $h^{-3}$ converts (the expectation value of) the occupation number $I(\eta, x^i; \vec{q}) \equiv \langle \hat{n}(x^i; \vec{q}) \rangle (\eta)$ to (the expectation value of) the number density $f(\eta, x^i; \vec{q})$ in phase space. (Actually the distribution function is often defined without the state density factor $h^3$, so that then $f = I$).

- We write the perturbed Stokes $I$ as

$$I(\eta, x^i; q, \hat{n}) = \bar{I}(\eta, q) + \delta I(\eta, x^i; q, \hat{n}) \equiv \frac{2}{\exp\left\{\frac{q}{T(\eta)\left[1 + \Theta(\eta, x^i; q, \hat{n})\right]}\right\} - 1} \tag{8}$$

just like in §F4, and note that for $1^{st}$ order perturbations

$$\delta I = \frac{\partial \bar{I}}{\partial T} \cdot T\Theta = -q \frac{\partial \bar{I}}{\partial q} \Theta \tag{9}$$

where $\bar{I}(\eta, q)$ is considered as a function of $T$ and $q$, $\bar{I}(T, q)$. As $\bar{I}$ depends on $\eta$ only through $T$, $\left(\frac{\partial \bar{I}}{\partial q}\right)_T = \left(\frac{\partial \bar{I}}{\partial q}\right)_\eta$. Here

$$\frac{\partial \bar{I}}{\partial q} = \frac{\partial}{\partial q}\left(\frac{2}{e^{q/T} - 1}\right) = \frac{-2}{(e^{q/T} - 1)^2} \cdot \frac{1}{T} e^{q/T} \tag{10}$$

and

$$\underline{-q \frac{\partial \bar{I}}{\partial q}} = \frac{2}{(e^{q/T} - 1)^2} \cdot \frac{q}{T} e^{q/T} = \frac{2x e^x}{(e^x - 1)^2} = \underline{\frac{2x}{(e^{x/2} - e^{-x/2})^2}} \tag{11}$$

where $x \equiv \frac{q}{T(\eta)}$. Thus

$$\delta I = \frac{2x}{(e^{x/2} - e^{-x/2})^2} \Theta \quad \Rightarrow \quad \underline{\Theta \equiv I_c = \frac{(e^{x/2} - e^{-x/2})^2}{2x} \delta I} \tag{12}$$

Here $I_c$ is the Stokes $I$ perturbation expressed as a relative CMB temperature perturbation. It is the same as the brightness function at §F4.

- We now redefine the other Stokes parameters also as relative perturbations to the CMB temperature,

$$\underline{\delta \rho^{(1)}_{ss'}} = \frac{1}{2}\begin{bmatrix} \delta I + Q & U - iV \\ U + iV & \delta I - Q \end{bmatrix} \qquad (13)$$

$$\equiv \underline{\left(-q\frac{\partial \bar{I}}{\partial q}\right)\cdot \rho_{ss'}} \equiv \underline{\left(-q\frac{\partial \bar{I}}{\partial q}\right)\cdot \frac{1}{2}\begin{bmatrix} \Theta + Q_c & U_c - iV_c \\ U_c + iV_c & \Theta - Q_c \end{bmatrix}}$$

where we defined the <u>CMB temperature perturbation density matrix</u>

$$\underline{\rho_{ss'}(\eta, x^i; q, \hat{n}) = \frac{1}{2}\begin{bmatrix} \Theta + Q_c & U_c - iV_c \\ U_c + iV_c & \Theta - Q_c \end{bmatrix}} \qquad (14)$$

so that

$$\underline{Q_c} \equiv \underline{\frac{1}{\left(-q\frac{\partial \bar{I}}{\partial q}\right)}} Q = \underline{\frac{\left(e^{x/2} - e^{-x/2}\right)^2}{2x}} Q \quad \text{etc.} \qquad (15)$$

- It will turn out, that in $1^{st}$ order perturbation theory <u>no V polarization ever develops</u>, so that $V_c = V = 0$. It will also turn out, that just like for $\Theta$, no photon energy (frequency) dependence ever develops for $Q_c$ and $U_c$, so that

$$\underline{\rho_{ss'}(\eta, x^i, \vec{q}) = \rho_{ss'}(\eta, x^i, \hat{n})} = \frac{1}{2}\begin{bmatrix} \Theta + Q_c & U_c \\ U_c & \Theta - Q_c \end{bmatrix}$$

Thus $\rho_{ss'}(\eta, x^i, \hat{n})$ is a real symmetric matrix, which represents <u>a time, location, direction, and polarization dependent perturbation in the CMB temperature</u>; while the frequency dependence of the CMB remains the black-body spectrum.

> $\Theta(\eta, x^i, \hat{n})$ is the temperature perturbation at $\eta, x^i$ of photons going in direction $\hat{n}$, averaged over the two polarization states
>
> $Q_c(\eta, x^i, \hat{n})$ is $\frac{1}{2}$ of the difference between the temperature perturbations of the photons in linear x-polarization state and in the linear y-polarization state
>
> $U_c(\eta, x^i, \hat{n})$ is the corresponding difference between the linear 45° and 135° polarization states.

- The reason they have no $q$ ($\nu$) dependence is the same as for $\Theta$. Initially they have none, as initially photons are in thermal equilibrium, so $Q = U = 0$. We shall derive evolution equations for $Q_c$ and $U_c$; and these evolution equations turn out to be independent of $q$ — thus no $q$-dependence develops, and the perturbations maintain the black-body spectrum.

## Perturbations in CMB Temperature and Occupation Number

- The two dimensionless ways of representing the CMB perturbation Stokes parameters can be related to each other as follows. Here

$$x = \frac{h\nu}{k_B T(\eta)} = \frac{q}{T(\eta)} \quad \text{in natural units.}$$

From Eqs. (P10.4, 12, 15)

$$\bar{I}(\vec{q}) = \frac{2}{e^x - 1}$$

$$\delta I(\vec{q}) = \underbrace{\frac{2x}{(e^{x/2} - e^{-x/2})^2}}_{-q\frac{\partial \bar{I}}{\partial q} = -x\frac{\partial \bar{I}}{\partial x}} \Theta \; ; \qquad Q(\vec{q}) = \frac{2x}{(e^{x/2} - e^{-x/2})^2} \Theta_c \qquad \text{etc.}$$

$$Q_c = \frac{1}{\left(-q\frac{\partial \bar{I}}{\partial q}\right)} Q(\vec{q}) \qquad \text{etc.}$$

Find $\bar{I}(\vec{q})$ and $-q\frac{\partial \bar{I}}{\partial q}$ for various $x$

| $x$ | $I(\vec{q})$ | $\left(-q\frac{\partial \bar{I}}{\partial q}\right)$ | $\left(-q\frac{\partial \bar{I}}{\partial q}\right)^{-1}$ |
|---|---|---|---|
| 0.1 | 19.016 663 89 | 19.983 341 66 | .050 041 680 57 |
| 0.2 | 9.033 311 132 | 9.966 733 227 | 0.100 333 7781 |
| 0.5 | 3.082 988 165 | 3.917 698 089 | 0.255 251 9304 |
| 1 | 1.163 953 414 | 1.841 347 188 | 0.543 080 6348 |
| 2 | 0.313 035 2855 | 0.724 061 6610 | 1.381 097 846 |
| 5 | .013 567 309 81 | .068 296 728 80 | 14.641 989 71 |
| 10 | $9.080\,398\,202 \times 10^{-5}$ | $9.080\,810\,470 \times 10^{-4}$ | 1101.223 292 |

$$\frac{\delta I(\vec{q})}{\Theta} = \frac{Q(\vec{q})}{Q_c} = \frac{U(\vec{q})}{U_c} \qquad \frac{\Theta}{\delta I(\vec{q})} = \frac{Q_c}{Q(\vec{q})} = \frac{U_c}{U(\vec{q})}$$

- From here on, we express the Stokes parameters as CMB temperature perturbation

$$\Theta(\eta, x^i, \hat{n})$$

$$Q_c(\eta, x^i, \hat{n}) \equiv Q(\eta, x^i, \hat{n})$$

$$U_c(\eta, x^i, \hat{n}) \equiv U(\eta, x^i, \hat{n})$$

and drop the subscript C we used for indicating this.