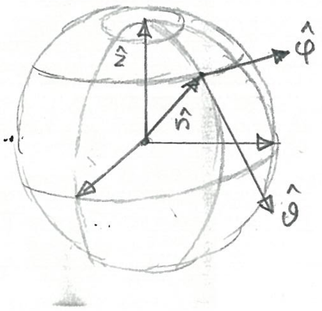
The Polarization Tensor

- Thus the CMB perturbations are described by the CMB temperature perturbation density matrix; which we can separate into its trace Θ and a traceless part:

$$\rho_{ss'}(\eta,\vec{x},\hat{n}) = \frac{1}{2}\begin{bmatrix}\Theta+Q & U \\ U & \Theta-Q\end{bmatrix} = \Theta\cdot\frac{1}{2}\begin{bmatrix}1 & \\ & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}Q & U \\ U & -Q\end{bmatrix}$$

- We now want to expand the $\vec{x}$-dependence of the polarization in plane waves (Fourier expansion) and the $\hat{n}$-dependence in spherical harmonics. But here we have to face the crd. dependence of Q and U. They are defined wrt some $\hat{x}$ and $\hat{y}$ directions that are orthogonal to the photon momentum direction $\hat{n}$. We need to make a choice how to determine the $\hat{x}$ and $\hat{y}$ for each $\hat{n}$.

- The set of unit vectors form a sphere, where we can use spherical crd's ϑ, φ. The unit vectors $\hat{\vartheta}, \hat{\varphi}, \hat{n}$ for an orthonormal right-handed basis. For each



Fourier mode we choose the spherical crd's so that the z-axis ($\vartheta=0$) is parallel to the Fourier wave vector $\vec{k}$. The polarization Stokes parameters for each photon direction $\hat{n}$ are then defined wrt to the $\hat{\vartheta}$ and $\hat{\varphi}$ directions at $\hat{n}$. (and not the $\hat{x}$ and $\hat{y}$, which are not orthogonal to $\hat{n}$. (except for $\hat{n} = \hat{k}$)).

Thus the two linear polarization states we consider are ϑ and φ linear polarization; $\hat{\vartheta}$ takes the former role of $\hat{x}$, and $\hat{\varphi}$ the former role of $\hat{y}$;

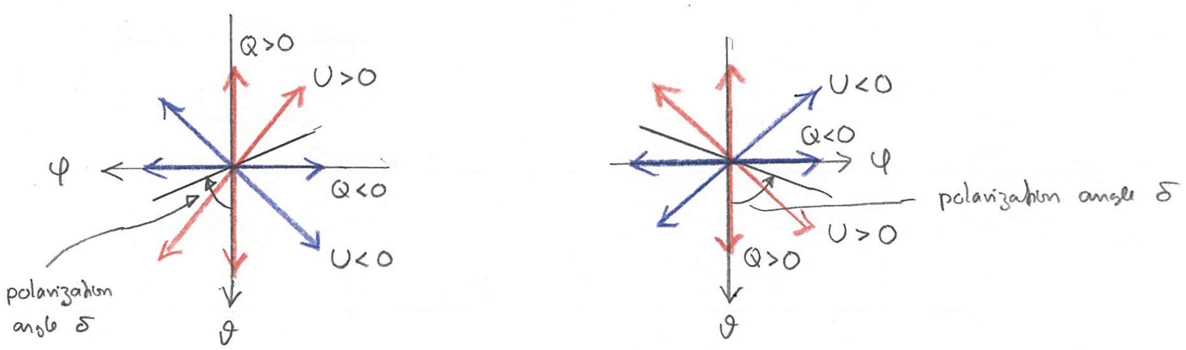


Fig. Looking at the unit sphere a) from the inside (photon going away from us)
b) from the outside (photon coming towards us)

- We know that $g_{\hat a \hat b}(\eta, \vec{x}, \hat{n}) = \frac{1}{2}\begin{bmatrix} \Theta + Q & U \\ U & \Theta - Q \end{bmatrix}$ transforms as a 2^{nd} order tensor in crd. transformations on the $\hat{n}$-sphere. We can separate it out into its trace,

  $\mathrm{Tr}\, g = \Theta$, which transforms as a scalar, and

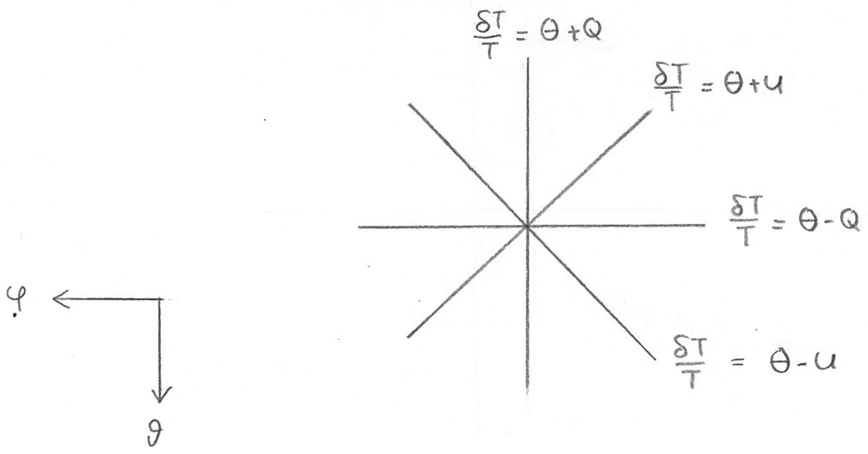
  the traceless symmetric polarization tensor $P_{\hat a \hat b} \equiv \frac{1}{2}\begin{bmatrix} Q & U \\ U & -Q \end{bmatrix}$

- At a given $\eta, \vec{x}$, $P_{\hat a \hat b}(\hat n)$ is a 2^{nd} order tensor field on a sphere. We put hats on the indices to indicate, that $P_{\hat a \hat b}$ are its components in an orthonormal basis $(\hat\vartheta, \hat\varphi)$; not in the ϑ, φ coordinate basis.

$$P_{\hat\vartheta\hat\vartheta} = -P_{\hat\varphi\hat\varphi} = \tfrac{1}{2} Q(\vartheta, \varphi)$$

$$P_{\hat\vartheta\hat\varphi} = P_{\hat\varphi\hat\vartheta} = \tfrac{1}{2} U(\vartheta, \varphi)$$

- As a tensor field on a sphere, the polarization tensor should not be expanded in ordinary spherical harmonics the same way as a scalar field, like Θ.[*] We shall now devote an entire chapter for the discussion of the spherical harmonic expansion of the polarization tensor field.



*) In a given crd. system, you could do it, if you wanted; but then the multipole coefficients of Q and U would have a different transformation law in crd. transformations from the case of scalar fields.