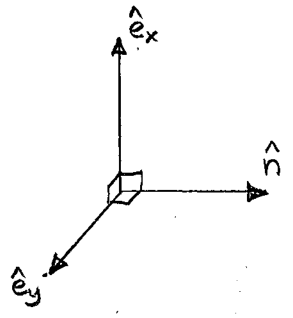
P. POLARIZATION

P1. Classical Monochromatic Plane Wave

- Consider a monochromatic plane wave propagating in the direction $\hat{n}$ with wavelength λ. The wave number is $q = \frac{2\pi}{\lambda}$, wave vector $\vec{q} = q\hat{n}$ ($=$ photon momentum $= \hbar\vec{q} = \vec{q}$). The electric field of this wave is

$$\underline{\vec{\mathcal{E}}(t,\vec{x}) = \mathrm{Re}\left[(E_x\hat{e}_x + E_y\hat{e}_y)\, e^{i\vec{q}\cdot\vec{x} - i\omega t}\right]} \qquad (1)$$



where $\omega = q = 2\pi\nu$

$$\underline{E_x = a_x e^{i\alpha_x}}, \quad \underline{E_y = a_y e^{i\alpha_y}} \qquad (2)$$

are complex numbers ($a_x, a_y, \alpha_x, \alpha_y$ are real) and

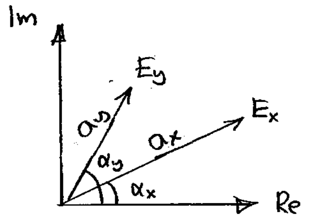
$\hat{e}_x, \hat{e}_y, \hat{n}$ form an orthonormal basis.(*

(That is, we chose a crd. system where $\hat{n} = \hat{z}$ $\Rightarrow$ $i\vec{q}\cdot\vec{x} = iqz = i\omega z$

Note $\vec{q} \neq \vec{k}$, where $\vec{k}$ is a perturbation Fourier mode wave vector; so at some point we need to rotate crd. systems.)

- The plane wave is thus completely described by the constant complex vector

$$\underline{\vec{E} \equiv E_x\hat{e}_x + E_y\hat{e}_y} \qquad (3)$$



and the wave vector $\vec{q}$.

$|E_x| = a_x, \quad |E_y| = a_y$

- At a given location, e.g. $\vec{x} = 0$

$$\vec{\mathcal{E}}(t) = \mathrm{Re}\left[(E_x\hat{e}_x + E_y\hat{e}_y)e^{-i\omega t}\right] = \mathrm{Re}\left[|E_x|e^{i(\alpha_x - \omega t)}\,\hat{e}_x + |E_y|e^{i(\alpha_y-\omega t)}\right]$$

$$= \underbrace{|E_x|\cos(\alpha_x - \omega t)}_{\mathcal{E}_x(t)}\,\hat{e}_x + \underbrace{|E_y|\cos(\alpha_y - \omega t)}_{\mathcal{E}_y(t)}\,\hat{e}_y$$

$$\underline{\alpha \equiv \alpha_y - \alpha_x}$$

We may choose the origin of time so that $\alpha_x = 0$

$$\Rightarrow \vec{\mathcal{E}}(t) = |E_x|\cos\omega t\;\hat{e}_x + |E_y|\cos(\omega t - \alpha)\,\hat{e}_y \qquad (4)$$

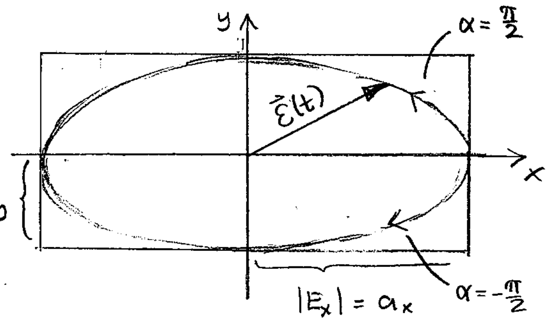
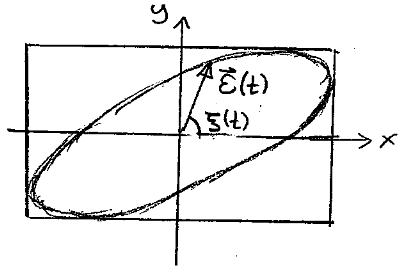


Fig. $\vec{\mathcal{E}}$ stays inside this box.

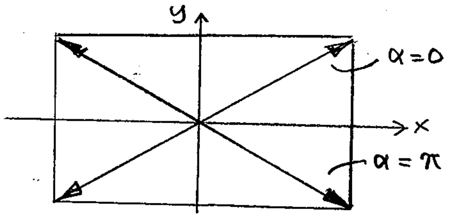
*) We use a right-handed basis. However, sometimes another right-handed basis $\{\hat{e}_x, \hat{e}_y, \underset{\sim}{\hat{n}}\}$ is used, where $\underset{\sim}{\hat{n}}$ is the direction the wave is coming from! This interchanges $\hat{e}_x$ and $\hat{e}_y$.
$= -\hat{n}$ (i.e. to which we are looking)

- In general, $\vec{\mathcal{E}}(t)$ draws an ellipse. The instantaneous components of the electric field are

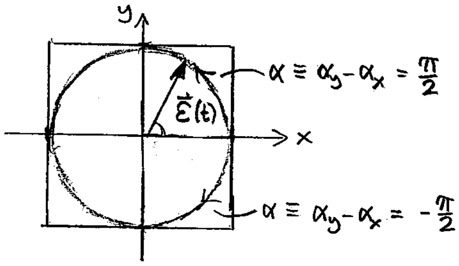


$$\mathcal{E}_x(t,\vec{x}) = |E_x|\cos(qz-\omega t+\alpha_x)$$
$$\mathcal{E}_y(t,\vec{x}) = |E_y|\cos(qz-\omega t+\alpha_y)$$

$|E_x|$ and $|E_y|$ give their maximum values (amplitudes).
(Note that the angle ξ does not rotate at a uniform rate, except for circular polarization.)



The cases $\alpha=0$ or $\alpha=\pi$ give linear polarization.



The cases $a_x=a_y$ and $\alpha=\pm\frac{\pi}{2}$ give circular polarization.

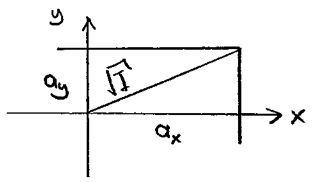
Stokes parameters

$$\boxed{\begin{aligned}
I &\equiv |E_x|^2+|E_y|^2 = a_x^2+a_y^2\\
Q &\equiv |E_x|^2-|E_y|^2 = a_x^2-a_y^2\\
U &\equiv E_xE_y^*+E_yE_x^* = 2\mathrm{Re}(E_xE_y^*) = 2a_xa_y\cos(\alpha_x-\alpha_y) = 2a_xa_y\cos(\overbrace{\alpha_y-\alpha_x}^{\alpha})\\
V &\equiv i(E_xE_y^*-E_yE_x^*) = 2\,\mathrm{Im}(E_x^*E_y) = -2a_xa_y\sin(\alpha_x-\alpha_y) = 2a_xa_y\sin(\underbrace{\alpha_y-\alpha_x}_{\alpha})
\end{aligned}} \qquad (5)$$

It is easy to show (exercise), that $\underline{I^2=Q^2+U^2+V^2}$ (6)

(a classical monochromatic plane wave is always completely polarized).

I is always positive, $Q,U,V\in[-I,I]$ can be positive, negative, or zero.

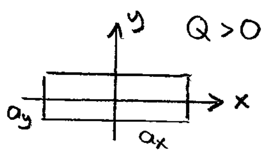


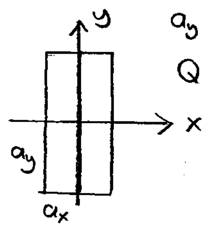
I gives the "size of the box", i.e. the half-diagonal is $\sqrt{I}$.

$a_x>a_y$
$Q>0$

$a_y>a_x$
$Q<0$

Q gives the "shape of the box":





$P\equiv\sqrt{Q^2+U^2} = 2a_xa_y$ is (half of the) "area of the box"

and the division of P into Q and U represents the phase difference $\alpha\equiv\alpha_y-\alpha_x$

- In SI units, the energy density of the electromagnetic wave is

$$\varrho = \tfrac{1}{2}\varepsilon_0 \mathcal{E}^2 + \frac{1}{2\mu_0}\mathcal{B}^2 = \varepsilon_0 \mathcal{E}^2 = \varepsilon_0 \vec{\mathcal{E}}(t)\cdot\vec{\mathcal{E}}(t)$$

(where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the electric and magnetic fields; for a plane wave $\mathcal{B} = \frac{1}{c}\mathcal{E}$ and $\vec{\mathcal{B}} \perp \vec{\mathcal{E}}$; both being in the xy-plane, i.e., $\perp \vec{q}$)

and the intensity is $\mathcal{I} = \varrho c = c\varepsilon_0 \mathcal{E}^2$. We get

$$\mathcal{E}^2 = \vec{\mathcal{E}}(t)\cdot\vec{\mathcal{E}}(t) = \mathcal{E}_x(t)^2 + \mathcal{E}_y(t)^2 = |E_x|^2\cos^2(\alpha_x - \omega t) + |E_y|^2\cos^2(\alpha_y - \omega t) \qquad (7)$$

whose time average (over one oscillation period) is

$$\langle \mathcal{E}^2 \rangle = \tfrac{1}{2}\left(|E_x|^2 + |E_y|^2\right) = \tfrac{1}{2} I \qquad (6)$$

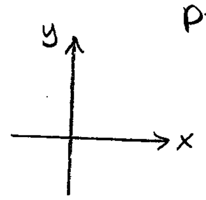
$\Rightarrow$ The intensity of the electromagnetic wave (in W/m²) is $\underline{\langle \mathcal{I}\rangle = \frac{1}{2}c\varepsilon_0 I}$ (8)

Thus the Stokes parameter <u>I is proportional to the intensity of the radiation.</u>

- In my units $\varepsilon_0 = \mu_0 = c = \hbar = k_B = 1$, so we get $\langle \mathcal{I}\rangle = \frac{1}{2}I$.

Thus it might have been better to include a $\sqrt{2}$ in the rhs of (1) or 1/2 on the rhs of (2) to make the Stokes parameters only half as big (as some authors do), so that I would be exactly the intensity of the radiation. However, I have followed what seems the most common practice. In general, there seems to be a lack of attention to the overall normalization of the Stokes parameters in the literature. So whenever you see or give the numerical values of Stokes parameters in some units, check what normalization is used.

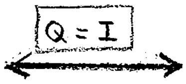
- In the CMB context, the Stokes parameters are redefined as relative to the intensity of background solution, and given either as dimensionless or in Kelvins, so this takes care of the normalization.

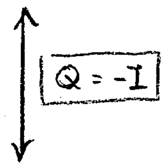


- The special cases where only one of Q, U, V is nonzero:

$\underline{Q = I} \Rightarrow |E_y| = 0$

$\underline{Q = -I} \Rightarrow |E_x| = 0$

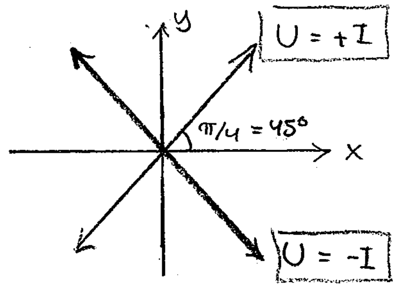




$\underline{U = \pm I} \Rightarrow Q = V = 0 \Rightarrow |E_x| = |E_y|$

and $\alpha = 0 \Rightarrow U = +I$

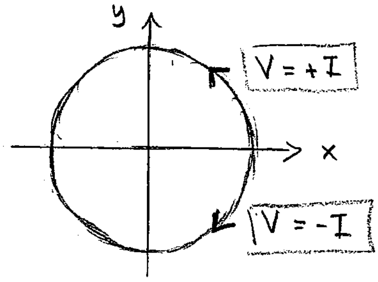
or $\alpha = \pi \Rightarrow U = -I$



$\underline{V = \pm I} \Rightarrow Q = U = 0 \Rightarrow |E_x| = |E_y|$

and $\alpha = \frac{\pi}{2} \Rightarrow V = +I$

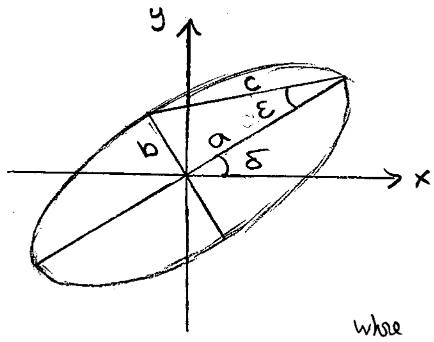
or $\alpha = -\frac{\pi}{2} \Rightarrow V = -I$



∴ Thus V represents circular polarization,

Q and U (and $P \equiv \sqrt{Q^2+U^2}$) represent linear polarization.

- Instead of the Stokes parameters, the monochromatic plane wave can also be described in terms of the ellipse the electric field vector $\vec{E}(t)$ draws in one period: its major and minor axes, a and b, and orientation angle δ. Instead of a and b,



we can use $c = \sqrt{a^2+b^2}$

and $\varepsilon = \arctan \frac{b}{a}$ (ellipticity angle)

we choose $\delta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\varepsilon \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

where $\varepsilon < 0$ $(b < 0)$ corresponds to clockwise rotation of $\vec{E}(t)$.

The Stokes parameters are related to these (exercise) by

$$\boxed{\begin{aligned} I &= c^2 \\ Q &= I \cos 2\varepsilon \cos 2\delta \\ U &= I \cos 2\varepsilon \sin 2\delta \\ V &= I \sin 2\varepsilon \end{aligned}} \qquad (9)$$

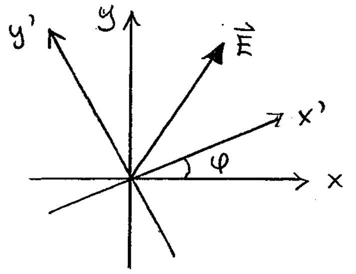
$$\Rightarrow \tan 2\delta = \frac{U}{Q} \Rightarrow \underline{\delta = \frac{1}{2} \arctan \frac{U}{Q}} \qquad (10)$$

$$P \equiv \sqrt{Q^2+U^2} = I \cos 2\varepsilon$$

$$\Rightarrow \underline{\varepsilon = \frac{1}{2} \arccos \frac{P}{I}} \qquad (11)$$

Rotation of coordinates

- The Stokes parameters are defined wrt the xy crd. system. If we rotate this system around the wave vector $\vec{q}$, the Stokes parameters get transformed. The components of the complex amplitude vector $\vec{E}$ transform in the usual manner:



$$\begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} = \begin{bmatrix} \cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \qquad (12)$$

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix}$$

From this we get (exercise):

$$\boxed{\begin{aligned} I' &\equiv |E_{x'}|^2 + |E_{y'}|^2 = \dots = |E_x|^2 + |E_y|^2 = I \\ \underline{Q'} &\equiv |E_{x'}|^2 - |E_{y'}|^2 = \dots = \underline{\cos 2\varphi \cdot Q + \sin 2\varphi \cdot U} \\ \underline{U'} &\equiv E_{x'} E_{y'}^* + E_{y'} E_{x'}^* = \dots = \underline{-\sin 2\varphi \cdot Q + \cos 2\varphi \cdot U} \\ V' &\equiv i\,(E_{x'} E_{y'}^* - E_{y'} E_{x'}^*) = \dots = V \end{aligned}} \qquad (13)$$

Thus $\underline{I \text{ and } V \text{ are invariant}}$, but Q and U transform into each other.

We also note that if we think of Q and U as components of a 2-dim "vector" $\tilde{p} \equiv \begin{bmatrix} Q \\ U \end{bmatrix}$

$$\begin{bmatrix} Q' \\ U' \end{bmatrix} = \begin{bmatrix} \cos 2\varphi & +\sin 2\varphi \\ -\sin 2\varphi & \cos 2\varphi \end{bmatrix} \begin{bmatrix} Q \\ U \end{bmatrix}$$

i.e., it "rotates" by angle 2φ in QU-space, with its "length"

$$\underline{p \equiv \sqrt{Q^2 + U^2} \quad \text{invariant.}} \qquad (14)$$

We can also define a linear polarization "vector" $\bar{p}$ in the xy-plane, with magnitude p and direction angle

$$\underline{\delta \equiv \tfrac{1}{2} \arctan \frac{U}{Q}}$$

wrt the x-axis. It gives the direction in which $\vec{\mathcal{E}}(t)$ oscillates if $V = 0$. Since $\bar{p}$ and $-\bar{p}$ describe the same oscillation, the direction of $\bar{p}$ is only defined mod π, and we choose

$$\delta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}],$$

the arctan branch is picked by $\underline{\operatorname{sign} \delta = \operatorname{sign} U}$

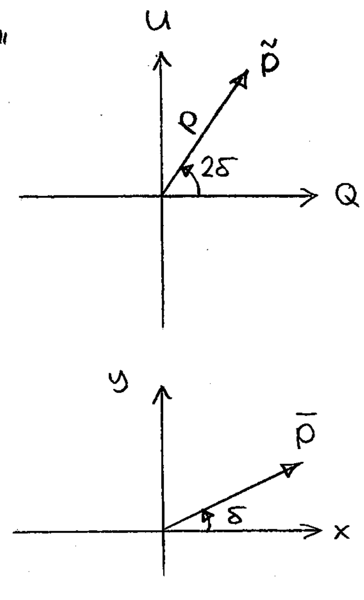


Fig. When the components of $\bar{p}$ rotate by φ in xy-space, $\tilde{p} = (Q, U)$ rotates by 2φ in QU-space.