L8. Approximate Results for C_l

- We can use the δ-function approximation of Sec. L5 to obtain qualitative understanding of the C_l.

- Inserting the δ-function approximations (5.4,6,7) for $\Theta_l^m(\eta_0,k)$ into (6.33) we get

$$
\begin{aligned}
C_l^{TT} &= \frac{4\pi}{(2l+1)^2}\sum_{m=-2}^{2}\int\frac{dk}{k}\left[\Theta_l^m(\eta_0,k)\right]^2 \mathcal{P}_m(k)\\
&\simeq 4\pi\int\frac{dk}{k}\mathcal{P}_0(k)\Big\{\left[\Theta_0^0(\eta_*,k)+\phi(\eta_*,k)\right]j_l(k\eta_0-k\eta_*) + v_b^{(0)}(\eta_*,k)\,j_l'(k\eta_0-k\eta_*)\\
&\qquad + P^{(0)}(\eta_*,k)\,j_l^{(20)}(k\eta_0-k\eta_*) + \int_{\eta_*}^{\eta_0} d\eta\,(\phi'+\psi')(\eta,k)\,j_l(k\eta_0-k\eta)\Big\}^2 \qquad \text{scalar contribution}\\
&+ 8\pi\int\frac{dk}{k}\mathcal{P}_1(k)\Big\{\left[v_b^{(1)}(\eta_*,k)-B^{(1)}(\eta_*,k)\right]j_l^{(11)}(k\eta_0-k\eta_*) + P^{(1)}(\eta_*,k)\,j_l^{(21)}(k\eta_0-k\eta_*)\\
&\qquad + \int_{\eta_*}^{\eta_0} d\eta\,\frac{k}{\sqrt{3}}B^{(1)}(\eta,k)\,j_l^{(21)}(k\eta_0-k\eta)\Big\}^2 \qquad \text{vector} \qquad (1)\\
&+ 8\pi\int\frac{dk}{k}\mathcal{P}_2(k)\Big\{P^{(2)}(\eta_*,k)\,j_l^{(22)}(k\eta_0-k\eta_*) - \int_{\eta_*}^{\eta_0} d\eta\, h^{(2)\prime}(\eta,k)\,j_l^{(20)}(k\eta_0-k\eta)\Big\}^2 \qquad \text{tensor}
\end{aligned}
$$

- For the polarization spectra we get (from Eqs. 5.8,9 and 6.34,35)

$$C_\ell^{EE} = \frac{4\pi}{(2\ell+1)^2}\sum_{m=-2}^{2}\int\frac{dk}{k}\left[E_\ell^m(\eta_0,k)\right]^2 \mathcal{P}_m(k)$$

$$\begin{aligned}
\simeq\ & 4\pi\cdot 6\int\frac{dk}{k}\mathcal{P}_0(k)\cdot\left[P^{(0)}(\eta_*,k)\right]^2 \mathcal{E}_\ell^0(k\eta_0-k\eta_*)^2 && \text{scalar}\\
&+8\pi\cdot 6\int\frac{dk}{k}\mathcal{P}_1(k)\cdot\left[P^{(1)}(\eta_*,k)\right]^2 \mathcal{E}_\ell^1(k\eta_0-k\eta_*)^2 && \text{vector}\\
&+8\pi\cdot 6\int\frac{dk}{k}\mathcal{P}_2(k)\cdot\left[P^{(2)}(\eta_*,k)\right]^2 \mathcal{E}_\ell^2(k\eta_0-k\eta_*)^2 && \text{tensor}
\end{aligned} \qquad (2)$$

$$C_\ell^{BB} \simeq \frac{4\pi}{(2\ell+1)^2}\sum_{m=-2}^{2}\int\frac{dk}{k}\left[B_\ell^m(\eta_0,k)\right]^2 \mathcal{P}_m(k)$$

$$\begin{aligned}
=\ & 8\pi\cdot 6\int\frac{dk}{k}\mathcal{P}_1(k)\cdot\left[P^{(1)}(\eta_*,k)\right]^2 \beta_\ell^1(k\eta_0-k\eta_*)^2 && \text{vector}\\
&+8\pi\cdot 6\int\frac{dk}{k}\mathcal{P}_2(k)\cdot\left[P^{(2)}(\eta_*,k)\right]^2 \beta_\ell^2(k\eta_0-k\eta_*)^2 && \text{tensor}
\end{aligned} \qquad (3)$$

- We see how in the δ-function approximation the polarization spectra pick their whole contribution from the last scattering surface $\eta=\eta_*$. The source there is the photon brightness and E-mode polarization quadrupole

$$P^{(m)}(\eta_*,k) = \frac{1}{10}\left[\Theta_2^m(\eta_*,k)-\sqrt{6}\,E_2^m(\eta_*,k)\right] \qquad (4)$$

- Since the source is the same for C_ℓ^{EE} and C_ℓ^{BB}, their relation is determined by the radial functions $\mathcal{E}_\ell^m$ and β_ℓ^m. The ratio of the contributions from a given scale k to C_ℓ^{BB} and C_ℓ^{EE} is given by

$$\frac{\beta_\ell^m(k\eta_0-k\eta_*)^2}{\mathcal{E}_\ell^m(k\eta_0-k\eta_*)^2} \qquad (5)$$

Inspection of the radial functions shows that for vector perturbations the B mode comes out stronger, while for tensor modes the E mode is slightly stronger (see Hu & White p. 601).

- For the cross spectrum we get, from (5.4,6,7,8) and (6.36),

$$
\begin{aligned}
C_\ell^{TE} &= -\frac{4\pi}{(2\ell+1)^2}\sum_{m=-2}^{2}\int\frac{dk}{k}\,\Theta_\ell^m(\eta_0,k)\,E_\ell^m(\eta_0,k)\,\mathcal{P}_m(k) \\
&\simeq +4\pi\cdot\sqrt{6}\int\frac{dk}{k}\,\mathcal{P}_0(k)\cdot P^{(0)}(\eta_*,k)\,\mathcal{E}_\ell^0(k\eta_0-k\eta_*)\cdot\Big\{\big[\Theta_0^0(\eta_*,k)+\phi(\eta_*,k)\big]j_\ell(k\eta_0-k\eta_*) \\
&\qquad + V_b^{(0)}(\eta_*,k)\,j_\ell'(k\eta_0-k\eta_*) + P^{(0)}(\eta_*,k)\,j_\ell^{(20)}(k\eta_0-k\eta_*) + \int_{\eta_*}^{\eta_0}d\eta\,(\phi'+\psi')(\eta,k)\,j_\ell(k\eta_0-k\eta)\Big\} \qquad \text{scalar} \\
&\quad + 8\pi\cdot\sqrt{6}\int\frac{dk}{k}\,\mathcal{P}_1(k)\cdot P^{(1)}(\eta_*,k)\,\mathcal{E}_\ell^1(k\eta_0-k\eta_*)\cdot\Big\{\big[V_b^{(1)}(\eta_*,k)-B^{(1)}(\eta_*,k)\big]j_\ell^{(11)}(k\eta_0-k\eta_*) \\
&\qquad + P^{(1)}(\eta_*,k)\,j_\ell^{(21)}(k\eta_0-k\eta_*) + \int_{\eta_*}^{\eta_0}d\eta\,\frac{k}{\sqrt{3}}B^{(1)}(\eta,k)\,j_\ell^{(21)}(k\eta_0-k\eta)\Big\} \qquad \text{vector} \\
&\quad + 8\pi\cdot\sqrt{6}\int\frac{dk}{k}\,\mathcal{P}_2(k)\cdot P^{(2)}(\eta_*,k)\,\mathcal{E}_\ell^2(k\eta_0-k\eta_*)\cdot\Big\{P^{(2)}(\eta_*,k)\,j_\ell^{(22)}(k\eta_0-k\eta_*) \\
&\qquad - \int_{\eta_*}^{\eta_0}d\eta\,h^{(2)\prime}(\eta,k)\,j_\ell^{(20)}(k\eta_0-k\eta)\Big\} \qquad \text{tensor}
\end{aligned} \tag{6}
$$