L7.5 Transfer Functions and the Calculation of C_ℓ.

- The Stokes parameters Θ, Q, U, the neutrino brightness function N, metric perturbations $h_{\mu\nu}$, and fluid perturbations, δ, $\vec{v}$, M_{ij} are all real quantities in $\vec{x}, \hat{n}$ space. Their Fourier coefficients and multipoles (and "spherical" components $V^{(\pm1)}$, $h^{(\pm2)}$ etc) are, however, complex.

- But we have cleverly chosen the phase conventions for the multipoles, and $V^{(m)}$ etc., so that no complex coefficients appear in our final perturbation equations (we deviated from this path for $B^{(\pm1)}$ and $V^{(\pm1)}$ on p. F7.2 and p. C7.2; but corrected their definitions on p. L1.1). This worthy goal explains the mysterious phase choices, e.g. the i^ℓ in $\Theta_\ell^m \equiv i^\ell \sqrt{\frac{2\ell+1}{4\pi}} a_{\ell m}$ (F5.5).

- This means that, if the initial Fourier coefficient $r_{k\ell}^m(0)$ of the primordial perturbations happened to be real; then everything would remain real as we evolve our eqs. In other words: <u>the transfer functions are real</u>.

- So, if you think of our final perturbation eqs. as eqs. for calculating the transfer functions (you start with $r_{k\ell}^m(0) = 1$); then no complex quantities appear in the calculation; you just evolve real quantities numerically.

- On the other hand, if you think of the eqs. as describing the evolution of the actual multipoles or Fourier coefficients of an actual universe; then all quantities appearing in our final perturbation evolution equations will maintain the same complex phase throughout the evolution, as the initial value $r_{k\ell}^m(0)$ of the mode in question had. The actual phase differences that exist between, e.g., density and velocity perturbations, when the fluid oscillates, are revealed when you convert back to "normal" Cartesian vector components, and to $\vec{x}$ space.

The (numerical) calculation of the C_ℓ consists of 4 steps:

1) For a given background universe (specified by some cosmological parameters), evolve the background equations, to obtain the background quantities:

$$a(\eta),\ \mathcal{H}(\eta),\ \bar{\rho}_i(\eta),\ \bar{p}_i(\eta)$$

that appear as coefficients in the perturbation eqs. You also need to do a (background) recombination calculation to obtain the H and He ionization fractions (also the He abundance must be specified as a cosmological parameter or as a BBN result to get this pinned down), to get

$$n_e(\eta),\ \tau(\eta),\ g(\eta)$$

2) For this background universe, choose a sufficiently densely spaced set of k values that cover the relevant scales; and for each of them evolve the perturbation eqs, using initial values $\Gamma^m_{\vec{k}}(0) = 1$, - i.e., you are calculating transfer functions, to obtain the evolution of the sources $\Theta^0_0(\eta,k)$, $\phi(\eta,k)$, $V^{(c)}_b(\eta,k)$, $\Theta^m_2(\eta,k)$, ..., all the quantities that appear as sources in the line-of-sight integrals (4.8-13). Although they only contain multipoles up to $L=2$; you need to include higher multipoles in the Boltzmann hierarchy, truncating the hierarchy cleverly at some $L = O(10)$. Thus no high multipoles need to be evolved.

3) Obtain from the transfer functions $\Theta^m_\ell(\eta_0,k)$, $E^m_L(\eta_0,k)$, $B^m_L(\eta_0,k)$ from the line-of-sight integrals (4.8-13). This gives you all the multipoles you are interested in, for $L \geq 2$.

4) Only at this stage you need to choose the primordial power spectra (amplitudes, spectral indices). The C_ℓ^{TT}, C_ℓ^{EE}, C_ℓ^{BB}, C_ℓ^{TE} are given by Eqs. (6.33-36). Note that steps 1-3 need to be done only once for a given background universe. You can then repeat step 4 many times for different spectral indices etc.