## L7.1 Energy Tensor Perturbations and Multipoles

- In the Einstein eqs the photon perturbations appear as the perturbations in the energy tensor: $\delta_\gamma$, $\vec{v}_\gamma$, $\Pi^{ij}_\gamma$. We need to relate these to the multipoles $\Theta_\ell^m$.

- From Eq. (F2.15) we have the relation of the photon distribution function $f$ to the energy tensor:

$$\rho_\gamma = \bar{\rho}_\gamma + \delta\rho_\gamma = \int q f \, d^3q \tag{2}$$

$$(\bar{\rho}_\gamma + \bar{p}_\gamma)\vec{v}_\gamma = \frac{4}{3}\bar{\rho}_\gamma \vec{v}_\gamma = \int q \hat{n} f \, d^3q \tag{3}$$

$$\Sigma^{ij}_\gamma = \bar{p}\,\Pi^{ij}_\gamma = \int (n^i n^j - \tfrac{1}{3}\delta^{ij})\, q f \, d^3q \tag{4}$$

- The photon distribution function is split into a homogeneous and isotropic background $\bar{f}$ and the perturbation $f^{(1)}$: $f(\eta,\vec{x},q,\hat{n}) = \bar{f}(\eta,q) + f^{(1)}(\eta,\vec{x},q,\hat{n})$. The perturbation is related to the brightness function $\Theta(\eta,\vec{x},\hat{n})$ by Eq. (F4.3):

$$f^{(1)} = -q\frac{\partial \bar{f}}{\partial q}\Theta \tag{5}$$

- We can now relate the photon energy tensor perturbations to the brightness function:

$$\delta\rho_\gamma = \int q f^{(1)} d^3q = -\int q^2 \frac{\partial \bar{f}}{\partial q}\Theta \cdot q^2 dq\, d\Omega = -\int q^4 \frac{\partial \bar{f}}{\partial q} dq \cdot \int \Theta(\hat{n})\, d\Omega \tag{6}$$

where partial integration gives

$$-\int q^4 \frac{\partial \bar{f}}{\partial q} dq = \int 4 q^3 \bar{f}\, dq = \frac{1}{\pi}\int q\bar{f} \cdot 4\pi q^2 dq = \frac{1}{\pi}\bar{\rho}_\gamma \tag{7}$$

Thus we have
$$\delta\rho_\gamma = \bar{\rho}_\gamma \cdot \frac{1}{\pi}\int \Theta(\hat{n})\, d\Omega \quad \Rightarrow \quad \underline{\delta_\gamma = \frac{1}{\pi}\int \Theta(\hat{n})\, d\Omega} \, . \tag{8}$$

- For velocity perturbations,

$$\frac{4}{3}\bar{\rho}_\gamma \vec{v}_\gamma = \int q\hat{n} f\, d^3q = \underbrace{\int q\hat{n}\bar{f} d^3q}_{=0,\ \text{since integrand is odd}} - \int q^2 \frac{\partial \bar{f}}{\partial q}\hat{n}\Theta \cdot q^2 dq\, d\Omega$$

$$= -\int q^4 \frac{\partial \bar{f}}{\partial q} dq \cdot \int \hat{n}\Theta(\hat{n})\, d\Omega = \frac{1}{\pi}\bar{\rho}_\gamma \cdot \int \hat{n}\Theta(\hat{n}) d\Omega \quad \Rightarrow \quad \vec{v}_\gamma = \frac{3}{4\pi}\int \hat{n}\Theta(\hat{n}) d\Omega \tag{9}$$

- For anisotropic stress, (the background part gives zero in integration)

$$\bar{p}_\gamma \Pi^{ij}_\gamma = \int (n^i n^j - \tfrac{1}{3}\delta_{ij})\, q f^{(1)} d^3q = -\int (n^i n^j - \tfrac{1}{3}\delta_{ij})\, q^2 \frac{\partial \bar{f}}{\partial q}\Theta \cdot q^2 dq\, d\Omega$$

$$= -\int q^4 \frac{\partial \bar{f}}{\partial q} dq \cdot \int (n^i n^j - \tfrac{1}{3}\delta_{ij})\Theta(\hat{n}) d\Omega = \frac{1}{\pi}\bar{\rho}_\gamma \cdot \int (n^i n^j - \tfrac{1}{3}\delta_{ij})\Theta(\hat{n}) d\Omega$$

Since $\bar{p}_\gamma = \frac{1}{3}\bar{\rho}_\gamma$, we have

$$\Pi^{ij}_\gamma = \frac{3}{\pi}\int (n^i n^j - \tfrac{1}{3}\delta_{ij})\Theta(\hat{n}) d\Omega \tag{10}$$

- Consider now a single Fourier mode $\vec{k}$ in the coordinates where $\hat{z} = \hat{k}$. We then have the multipoles $\Theta_l^m$ as (Eq. F5.5)

$$\Theta_l^m \equiv i^l \sqrt{\frac{2l+1}{4\pi}}\, a_{lm} \equiv i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega\, Y_l^{m*}(\hat{n})\Theta(\hat{n}) \tag{11}$$

Using Eqs. (F6.6) and (F6.7) we get (exercise)

$$\Theta_0^0 = \frac{1}{4\pi}\int \Theta(\hat{n}) d\Omega = \frac{1}{4}\delta_\gamma \tag{12}$$

$$\Theta_1^0 = i\frac{3}{4\pi}\int n_3 \Theta(\hat{n}) d\Omega = i v_\gamma^3 \equiv v_\gamma^{(0)} \tag{13}$$

$$\Theta_1^{\pm 1} = \mp i \frac{3}{4\pi}\int \frac{1}{\sqrt{2}}(n_1 \pm i n_2)\Theta(\hat{n}) d\Omega = \mp \frac{i}{\sqrt{2}}(v_\gamma^1 \mp i v_\gamma^2) \equiv v_\gamma^{(\pm 1)} \tag{14}$$

$$\Theta_2^0 = -\frac{15}{8\pi}\int (n_3^2 - \tfrac{1}{3})\Theta(\hat{n}) d\Omega = -\frac{5}{8}\Pi_\gamma^{33} \tag{15}$$

$$\Theta_2^{\pm 1} = \pm\sqrt{\frac{3}{2}}\cdot\frac{5}{4\pi}\int (n_3 n_1 \mp i n_3 n_2)\Theta(\hat{n}) d\Omega = \pm\sqrt{\frac{3}{2}}\frac{5}{12}\left(\Pi_\gamma^{13} \mp i\Pi_\gamma^{23}\right) \tag{16}$$

$$\Theta_2^{\pm 2} = -\sqrt{\frac{3}{2}}\cdot\frac{5}{8\pi}\int (n_{11} - n_{22} \mp 2i n_1 n_2)\Theta(\hat{n}) d\Omega = -\frac{1}{\sqrt{6}}\cdot\frac{5}{8}\left(\Pi_\gamma^{11} - \Pi_\gamma^{22} \mp 2i\Pi_\gamma^{12}\right) \tag{17}$$

- In the $\hat{z} = \hat{k}$ crd. system the anisotropic stress $\Pi^{ij}$ of the Fourier mode $\vec{k}$ is split into scalar, vector, and tensor parts as (see CDT §6 and §9)

$$\Pi_{ij} = \begin{bmatrix} \frac{1}{3}\Pi + \Pi^T_{11} & \Pi^T_{12} & \Pi^V_{13} \\ \Pi^T_{12} & \frac{1}{3}\Pi - \Pi^T_{11} & \Pi^V_{23} \\ \Pi^V_{13} & \Pi^V_{23} & -\frac{2}{3}\Pi \end{bmatrix} \tag{18}$$

So we can write Eqs. (15-17) as

$$\begin{aligned} \Theta_2^0 &= \frac{5}{12}\Pi_\gamma \\ \Theta_2^{\pm 1} &= \pm\sqrt{\frac{3}{2}}\,\frac{5}{12}\left(\Pi^V_{13} \mp i\,\Pi^V_{23}\right) \\ \Theta_2^{\pm 2} &= -\frac{1}{\sqrt{6}}\cdot\frac{5}{8}\left(2\Pi^T_{11} \mp 2i\,\Pi^T_{12}\right) = -\frac{1}{\sqrt{6}}\cdot\frac{5}{4}\left(\Pi^T_{11} \mp i\,\Pi^T_{12}\right) \end{aligned} \tag{19}$$

which can easily be inverted to give $\Pi_{ij}$ in terms of the $\Theta_2^m$ :

$$\begin{aligned} \Pi &= \frac{12}{5}\Theta_2^0 \\ \Pi^V_{13} &= \sqrt{\frac{2}{3}}\,\frac{6}{5}\left(\Theta_2^{+1} - \Theta_2^{-1}\right) \qquad & \Pi^V_{13} \mp i\,\Pi^V_{23} &= \pm\sqrt{\frac{2}{3}}\,\frac{12}{5}\Theta_2^{\pm 1} \\ \Pi^V_{23} &= i\sqrt{\frac{2}{3}}\,\frac{6}{5}\left(\Theta_2^{+1} + \Theta_2^{-1}\right) \\ \Pi^T_{11} &= -\frac{2\sqrt{6}}{5}\left(\Theta_2^{+2} + \Theta_2^{-2}\right) \qquad & \Pi^T_{11} \mp i\,\Pi^T_{12} &= -\frac{4\sqrt{6}}{5}\Theta_2^{\pm 2} \\ \Pi^T_{12} &= -i\,\frac{2\sqrt{6}}{5}\left(\Theta_2^{+2} - \Theta_2^{-2}\right) \end{aligned} \tag{20}$$