L6.4 Angular Power Spectra

- We can now, finally, do the (theoretical, i.e., expected) angular power spectra.

- $C_L^{TT} \equiv \langle a^T_{Lm} a^{T*}_{Lm} \rangle$ (should be independent of m)

$$\overset{(2)}{=} \sum_{\vec{k}\vec{k}'} \langle a^T_{\vec{k}Lm} a^{T*}_{\vec{k}'Lm} \rangle \overset{(7)}{=} \frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D^L_{mm'}(\vec{k}) D^L_{mm''}(\vec{k}')^* \langle \Theta^{m'}_{L\vec{k}}(\eta_0) \Theta^{m''}_{L\vec{k}'}(\eta_0)^* \rangle$$

$$\overset{(25)}{=} \frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D^L_{mm'}(\vec{k}) D^L_{mm''}(\vec{k}')^* \Theta^{m'}_L(\eta_0,k) \Theta^{m''}_L(\eta_0,k')^* \underbrace{\langle r^{m'}_{\vec{k}}(0)\, r^{m''}_{\vec{k}'}(0)^* \rangle}_{\delta_{\vec{k}\vec{k}'} \delta_{m'm''} \left(\frac{2\pi}{L}\right)^3 \frac{1}{4\pi k^3} \mathcal{P}_{m'}(k)}$$

$$\overset{(22)}{=} \frac{4\pi}{2L+1} \sum_{m'} \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{k}} \frac{1}{4\pi k^3} |D^L_{mm'}(\vec{k})|^2 \cdot |\Theta^{m'}_L(\eta_0,k)|^2 \cdot \mathcal{P}_{m'}(k) \tag{27}$$

- The discrete Fourier transform is converted to a continuous Fourier transform with the recipe (Cosmology II, Sec. 11.5.1)

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{k}} \rightarrow \int d^3k \tag{28}$$

- The Wigner D-function $D^L_{mm'}(\vec{k})$ is

$$D^L_{mm'}(\vec{k}) = D^L_{mm'}(\alpha,\beta,\gamma) = e^{-im\alpha} d^L_{mm'}(\beta) e^{-im'\gamma} \tag{29}$$

where $\beta \equiv \vartheta$ is the angle between $\vec{k}$ and the z-axis of the common crd. system.

$$\Rightarrow \quad \underline{|D^L_{mm'}(\vec{k})|^2 = d^L_{mm'}(\vartheta)^2} \tag{30}$$

- Wigner D-functions have an orthogonality relation (VMK p. 95, Eq. (5))

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\cos\beta \int_0^{2\pi} d\gamma \; D^{L_1}_{m_1m_1'}(\alpha,\beta,\gamma) D^{L_2}_{m_2m_2'}(\alpha,\beta,\gamma)^* = \frac{8\pi^2}{2L_1+1} \delta_{L_1L_2} \delta_{m_1m_2} \delta_{m_1'm_2'} \tag{31}$$

$(m_1 = m_2,\ m_1' = m_2')$

$$\Rightarrow \quad \underline{\int_0^{\pi} d\cos\vartheta \; d^L_{mm'}(\vartheta) d^{L'}_{mm'}(\vartheta) = \frac{2}{2L+1} \delta_{LL'}} \tag{32}$$

- Using (28) and (30), Eq. (27) becomes

$$\underline{C_L^{TT}} = \frac{4\pi}{2L+1} \sum_{m'} \int \frac{d^3k}{4\pi k^3}\, d^L_{mm'}(\theta)^2 \cdot |\Theta_L^{m'}(\eta_0,k)|^2 \cdot \mathcal{P}_{m'}(k)$$

$$= \frac{4\pi}{2L+1} \sum_{m'} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_{-1}^{1} d\cos\theta \cdot d^L_{mm'}(\theta)^2}_{\frac{2}{2L+1}} \cdot \int \frac{k^2 dk}{4\pi k^3} |\Theta_L^{m'}(\eta_0,k)|^2 \cdot \mathcal{P}_{m'}(k)$$

$$\underline{= \frac{4\pi}{(2L+1)^2} \sum_{m'=-2}^{2} \int \frac{dk}{k} |\Theta_L^{m'}(\eta_0,k)|^2 \cdot \mathcal{P}_{m'}(k)} \qquad (33)$$

which is, indeed, independent of m. In the m' sum, $\pm m'$ terms are equal.

- The extra factor $\frac{1}{2L+1}$ in Eq. (33) compared to Eq. (27) comes from the Wigner D-functions distributing equally the m' power in $\Theta_L^{m'}$ to all m (there are $2L+1$ of them).

- The polarization angular power spectra are obtained likewise:

$$\underline{C_L^{EE}} \equiv \langle a^E_{Lm}\, a^{E*}_{Lm} \rangle \overset{(10)}{=} \sum_{\vec{k}\vec{k}'} \langle a^E_{\vec{k}Lm}\, a^{E\,*}_{\vec{k}'Lm} \rangle$$

$$\overset{(17)}{=} \frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D^L_{mm'}(\hat{k})\, D^L_{mm''}(\hat{k}')^* \langle E^{m'}_{L\vec{k}}(\eta_0)\, E^{m''}_{L\vec{k}'}(\eta_0)^* \rangle$$

$$\overset{(25)}{=} \frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D^L_{mm'}(\hat{k})\, D^L_{mm''}(\hat{k}')^*\, E_L^{m'}(\eta_0,k)\, E_L^{m''}(\eta_0,k')^* \overbrace{\langle r^{m'}_{\vec{k}}(0)\, r^{m''}_{\vec{k}'}(0)^* \rangle}^{\delta_{\vec{k}\vec{k}'}\, \delta_{m'm''} \left(\frac{2\pi}{L}\right)^3 \frac{1}{4\pi k^3} \mathcal{P}_{m'}(k)}$$

$$\overset{(22)}{=} \frac{4\pi}{2L+1} \sum_{m'} \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{k}} \frac{1}{4\pi k^3} \left| D^L_{mm'}(\hat{k}) \right|^2 \cdot \left| E_L^{m'}(\eta_0,k) \right|^2 \cdot \mathcal{P}_{m'}(k)$$

$$\overset{(28,30)}{\rightarrow} \frac{4\pi}{2L+1} \sum_{m'} \underbrace{\int \frac{d^3k}{4\pi k^3}\, d^L_{mm'}(\theta)^2}_{\frac{1}{2L+1} \int \frac{dk}{k}} \cdot \left| E_L^{m'}(\eta_0,k) \right|^2 \cdot \mathcal{P}_{m'}(k)$$

$$\overset{(32)}{=} \underline{\frac{4\pi}{(2L+1)^2} \sum_{m'=-2}^{2} \int \frac{dk}{k} \left| E_L^{m'}(\eta_0,k) \right|^2 \cdot \mathcal{P}_{m'}(k)} \qquad (34)$$

$$\underline{C_L^{BB}} \equiv \langle a^B_{Lm}\, a^{B*}_{Lm} \rangle = \sum_{\vec{k}\vec{k}'} \langle a^B_{\vec{k}Lm}\, a^{B\,*}_{\vec{k}'Lm} \rangle = \dots$$

$$= \frac{4\pi}{(2L+1)^2} \sum_{m'=-2}^{2} \int \frac{dk}{k} \left| B_L^{m'}(\eta_0,k) \right|^2 \cdot \mathcal{P}_{m'}(k)$$

$$= \underline{\frac{8\pi}{(2L+1)^2} \sum_{m'=1}^{2} \int \frac{dk}{k} \left| B_L^{m'}(\eta_0,k) \right|^2 \cdot \mathcal{P}_{m'}(k)} \qquad (35)$$

since $B_L^0(\eta_0,k) = 0$; $\left| B_L^{m'}(\eta_0,k) \right| = \left| B_L^{-m'}(\eta_0,k) \right|$; and $\mathcal{P}_{-m'}(k) = \mathcal{P}_{m'}(k)$

(CMB evolution eqs.) (assumption of parity conservation)

- For temperature-polarization correlation angular power spectra, one has to take into account the different signs in Eqs. (7) and (17):

$$\underline{C_L^{TE}} \equiv \langle a_{Lm}^T a_{Lm}^{E\,*} \rangle = \sum_{\vec{k}\vec{k}'} \langle a_{\vec{k}Lm}^T a_{\vec{k}'Lm}^{E\,*} \rangle$$

$$= -\frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D_{mm'}^L(\vec{k}) D_{mm''}^L(\vec{k}')^* \langle \Theta_{L\vec{k}}^{m'}(\eta_0) E_{L\vec{k}'}^{m''}(\eta_0)^* \rangle$$

$$= -\frac{4\pi}{2L+1} \sum_{m'} \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{k}} \frac{1}{4\pi k^3} |D_{mm'}^L(\vec{k})|^2 \cdot \Theta_L^{m'}(\eta_0,k) E_L^{m'}(\eta_0,k)^* \cdot \mathcal{P}_{m'}(k)$$

$$= \underline{-\frac{4\pi}{(2L+1)^2} \sum_{m'=-2}^{2} \int \frac{dk}{k} \, \Theta_L^{m'}(\eta_0,k) E_L^{m'}(\eta_0,k)^* \cdot \mathcal{P}_{m'}(k)} \quad \text{(note sign!)} \qquad (36)$$

$$C_L^{TB} \equiv \langle a_{Lm}^T a_{Lm}^{B\,*} \rangle = +\frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D_{mm'}^L(\vec{k}) D_{mm''}^L(\vec{k}')^* \langle \Theta_{L\vec{k}}^{m'}(\eta_0) B_{L\vec{k}'}^{m''}(\eta_0)^* \rangle$$

$$= +\frac{4\pi}{(2L+1)^2} \sum_{m'=-2}^{2} \int \frac{dk}{k} \, \Theta_L^{m'}(\eta_0,k) B_L^{m'}(\eta_0,k)^* \cdot \mathcal{P}_{m'}(k) \qquad (37)$$

$$C_L^{EB} \equiv \langle a_{Lm}^E a_{Lm}^{B\,*} \rangle = -\frac{4\pi}{2L+1} \sum_{\vec{k}\vec{k}'} \sum_{m'm''} D_{mm'}^L(\vec{k}) D_{mm''}^L(\vec{k}')^* \langle E_{L\vec{k}}^{m'}(\eta_0) B_{L\vec{k}'}^{m''}(\eta_0)^* \rangle$$

$$= -\frac{4\pi}{(2L+1)^2} \sum_{m'=-2}^{2} \int \frac{dk}{k} \, E_L^{m'}(\eta_0,k) B_L^{m'}(\eta_0,k)^* \cdot \mathcal{P}_{m'}(k) \qquad (38)$$

- <u>If</u> we assume that the random process, that generated the primordial perturbations, was <u>parity conserving</u> $\Rightarrow \mathcal{P}_{-m'}(k) = \mathcal{P}_{m'}(k)$

<u>then $C_L^{TB} = C_L^{EB} = 0$,</u> (39)

as the $m' = -1, -2$ terms cancel the $m' = +1, +2$ terms, since $B_L^{-m'}(\eta_0,k) = -B_L^{m'}(\eta_0,k)$; and the $m'=0$ term vanishes, since $B_L^0(\eta_0,k) = 0$.