L6.2 Primordial Power Spectra

- The values for the multipoles today, $\Theta^m_{\ell k}(\eta_0)$, $E^m_{\ell k}(\eta_0)$, $B^m_{\ell k}(\eta_0)$ depend on initial conditions linearly through the evolution described by the Boltzmann equations.

- For CMB physics, the initial conditions are usually specified during the "early radiation dominated epoch" (= the "initial epoch"). This corresponds to the era (*) when 1) all cosmologically relevant scales ($\gtrsim 1$ Mpc comoving) were well outside the horizon, 2) photons and baryons were tightly coupled and formed a single perfect fluid ($v_\gamma = v_b$; and the $\ell \geq 2$ photon multipoles were zero), 3) the universe was radiation dominated, but 4) neutrinos have already decoupled and electron-positron annihilation had taken place. (*)

- For simplicity, we shall consider only adiabatic initial conditions. This means that the perturbations for all fluid components are "the same", i.e.,

$$
\begin{aligned}
V_c &= V_b = V_\gamma = V_\nu \\
\delta_c &= \delta_b = \tfrac{3}{4}\delta_\gamma = \tfrac{3}{4}\delta_\nu
\end{aligned}
\tag{18}
$$

(the factor $\frac{3}{4}$ comes from the different eq. state of photons and neutrinos; adiabatic perturbations correspond to equal relative perturbations in the number density of each species).

- More complicated, nonadiabatic, initial conditions ("entropy perturbations", or "isocurvature modes") for scalar perturbations, were discussed in the 2004 CMB Physics course.

- For adiabatic perturbations we can describe the initial conditions for each $m, \vec{k}$ mode by a single quantity; which we shall generically denote by $r^m_{\vec{k}}(0)$, where $m = 0, \pm 1, \pm 2$ corresponding to scalar, vector, and tensor perturbations.

  ("r" here stands for "random", since we shall assume the initial conditions were produced by some random process. "0" stands for "initial time", although this is not really $\eta = 0$ — we could have written $r^m_{\vec{k}}(\eta_{init})$ as well. There may be some evolution, e.g. inflation + reheating + early rad.dom evolution, before η_{init}, responsible for the values $r^m_{\vec{k}}(\eta_{init}) \equiv r^m_{\vec{k}}(0)$; but it is not the topic of this course.)

*) Some time after BBN.

- There is some freedom in what physical quantities we chose as $r^m_{\vec{k}}$, to represent the initial conditions. A requirement is that they must me such quantities that remain constant during the "initial epoch". The various possibilities are related to each other by constant factors. We choose:

For scalar perturbations: $r^0_{\vec{k}} = \mathcal{R}_{\vec{k}}$, the comoving curvature perturbation (19)

For tensor perturbations: $r^{\pm 2}_{\vec{k}} = \sqrt{3}\, h^{(\pm 2)}_{\vec{k}} = -\frac{1}{\sqrt{2}}\left[h_+(\vec{k}) \mp i h_-(\vec{k})\right]$, (20)

the metric tensor perturbation

The factor $\sqrt{3}$ for tensor perturbations is so that we match the normalization of the primordial tensor perturbation power spectrum at Chapter T.

- From the discussion of the physics during the "initial epoch" in 2004 CMB Physics (Section §I1), we have the relations between the various constant physical quantities that could be used to specify adiabatic initial quantities for scalar perturbations

$$\delta_c = \delta_b = -\tfrac{3}{2}\phi \qquad \delta_\nu = \delta_\gamma = -2\phi \qquad (21)$$
$$\psi = \left(1+\tfrac{2}{5}f_\nu\right)\phi \qquad \mathcal{R} = -\tfrac{3}{2}\left(1+\tfrac{4}{15}f_\nu\right)\phi$$

where $f_\nu = \dfrac{\bar{\rho}_\nu}{\bar{\rho}_\nu + \bar{\rho}_\gamma} = \text{const} \sim 0.405$.

- We have not studied the physics of vector perturbations, but presumably suitable quantities related to the vector parts of the common velocity perturbation $V_c = V_b = V_\gamma = V_\nu$, or the shift vector, that remain constant during the "initial epoch", can be found to represent the "initial conditions" for vector perturbations as $r^{\pm 1}_{\vec{k}}$.

- We assume the primordial perturbations $r^m_{\vec{k}}(0)$ are the result of some isotropic Gaussian random process. Gaussian $\Rightarrow$ different Fourier and m modes $r^m_{\vec{k}}(0)$ are indepent random variables $\Rightarrow$ $\langle r^m_{\vec{k}}(0) r^{m'}_{\vec{k}'}(0)^* \rangle = 0$ for $m \neq m'$ or $\vec{k} \neq \vec{k}'$.

Isotropic $\Rightarrow$ $\langle r^m_{\vec{k}}(0) r^m_{\vec{k}}(0)^* \rangle$ depends only on the magnitude k of $\vec{k}$, not direction.

$$\therefore \quad \boxed{\langle r^m_{\vec{k}}(0)\, r^{m'}_{\vec{k}'}(0)^* \rangle = \delta_{\vec{k}\vec{k}'} \delta_{mm'} \left(\frac{2\pi}{L}\right)^3 \frac{1}{4\pi k^3} \mathcal{P}_m(k)} \tag{22}$$

where $\mathcal{P}_m(k)$ is the primordial power spectrum.(*

- We assume also that the <u>random process is parity conserving</u> $\Rightarrow$ $\underline{\mathcal{P}_{-m}(k) = \mathcal{P}_m(k)}$. (23)

Thus we have, assuming adiabatic primordial perturbations, three primordial power spectra:

$\mathcal{P}_0(k)$	scalar perturbations
$\mathcal{P}_1(k)$	vector perturbations
$\mathcal{P}_2(k)$	tensor perturbations

*) The factor $\frac{1}{L^3}$ is related to the use of a reference volume $V = L^3$ for a discrete Fourier transform (series instead of an integral). The factor

$$\frac{(2\pi)^3}{4\pi k^3} = \frac{2\pi^2}{k^3} \tag{24}$$

is related to the number of different Fourier modes $\vec{k}$ corresponding to a logarithmic interval $d \ln k$, and is included so that $\mathcal{P}_m(k)$ directly gives the perturbation power per logarithmic interval in scale.

The complex conjugate in (22) guarantees that $\mathcal{P}_m(k)$ is real (and nonnegative).