L6. Angular Power Spectra

L6.1 Inversion and Rotation

- Having the $\Theta_\ell^m(\eta_0)$, $E_\ell^m(\eta_0)$, $B_\ell^m(\eta_0)$ for each Fourier mode $\vec{k}$ allows us to calculate the observed angular power spectra C_ℓ^{TT}, C_ℓ^{TE}, C_ℓ^{EE}, C_ℓ^{BB}.

- There are two issues we need to take into account:

  1) The multipoles $\Theta_\ell^m(\eta)$, $E_\ell^m(\eta)$, $B_\ell^m(\eta)$ are for an expansion of $\Theta(\eta,\vec{x},\hat{n})$, $(Q\pm iU)(\eta,\vec{x},\hat{n})$, where $\hat{n} = \hat{n}_{phot}$ is the photon momentum direction.

     But we want the C_ℓ of the observed sky $\Theta_{obs}(\hat{n})$, $(Q\pm iU)_{obs}(\hat{n})$, where $\hat{n} = \hat{n}_{obs} = -\hat{n}_{phot}$

  2) For each Fourier mode $\vec{k}$, the multipoles $\overbrace{\Theta_\ell^m, E_\ell^m, B_\ell^m}^{\text{denote those by } \Theta_{\ell\vec{k}}^m,\, E_{\ell\vec{k}}^m,\, B_{\ell\vec{k}}^m}$ need to be rotated from the $\vec{k}$-related $(\hat{z}=\hat{k})$ crd. system to a common spherical crd. system (e.g. galactic or ecliptic crd's) before we can add the $\vec{k}$ modes up.

- The observed temperature anisotropy is

choose $\vec{x}_{obs} = 0 \hookrightarrow = 1$

$$\Theta_{obs}(\hat{n}) = \sum a_{\ell m}^T Y_\ell^m(\hat{n}) = \Theta(\eta_0,\vec{x}_{obs},-\hat{n}) = \sum_{\vec{k}} \Theta_{\vec{k}}(\eta_0,-\hat{n})\, \overbrace{e^{i\vec{k}\cdot\vec{x}_{obs}}}^{=1} \tag{1}$$

$$\Rightarrow \quad a_{\ell m}^T = \int d\Omega\, Y_\ell^{m*}(\hat{n})\,\Theta_{obs}(\hat{n}) = \sum_{\vec{k}} \int d\Omega\, Y_\ell^{m*}(\hat{n})\,\Theta_{\vec{k}}(\eta_0,-\hat{n}) \equiv \sum_{\vec{k}} a_{\vec{k}\ell m}^T \tag{2}$$

since integrate over all $\hat{n}$

$$\text{where} \quad a_{\vec{k}\ell m}^T \equiv \int d\Omega\, Y_\ell^{m*}(\hat{n})\,\Theta_{\vec{k}}(\eta_0,-\hat{n}) = \int d\Omega\, \underbrace{Y_\ell^{m*}(-\hat{n})}_{(-1)^\ell Y_\ell^{m*}(\hat{n}) \quad (\text{Eq. Y4.10})}\Theta_{\vec{k}}(\eta_0,\hat{n})$$

$$= (-1)^\ell \int d\Omega\, Y_\ell^{m*}(\hat{n})\,\Theta_{\vec{k}}(\eta_0,\hat{n}) \tag{3}$$

is the $\vec{k}$ mode contribution to the multipole $a_{\ell m}^T$ of the observed sky in the common spherical crd. system.

- Denote by $\tilde{a}^T_{\vec{k}\ell m}$ the corresponding multipole in the $\hat{z}=\hat{k}$ crd. system.

$$\tilde{a}^T_{\vec{k}\ell m} = (-1)^\ell \int d\Omega\, Y_\ell^{m*}(\hat{n})\,\Theta_{\vec{k}}(\eta_0,\hat{n}) \qquad \text{same expression, different crd. system !}$$

From Eq. (F5.5)

$$\Theta^m_{\ell\vec{k}}(\eta_0) \equiv i^\ell \sqrt{\frac{2\ell+1}{4\pi}} \int d\Omega\, Y_\ell^{m*}(\hat{n})\,\Theta_{\vec{k}}(\eta_0,\hat{n}) = (-i)^\ell \sqrt{\frac{2\ell+1}{4\pi}}\,\tilde{a}^T_{\vec{k}\ell m} \tag{4}$$

$$\Rightarrow \quad \tilde{a}^T_{\vec{k}\ell m} = i^\ell \sqrt{\frac{4\pi}{2\ell+1}}\,\Theta^m_{\ell\vec{k}}(\eta_0) \tag{5}$$

In the notation of this Chapter, $a_{\ell m}$ refer to multipoles in terms of $\hat{n} = \hat{n}_{obs}$, whereas Θ^m_ℓ, E^m_ℓ, B^m_ℓ to multipoles in terms of $\hat{n} = \hat{n}_{phot} = -\hat{n}_{obs}$.

Therefore we get some sign differences in the $a_{\ell m} \Leftrightarrow \Theta^m_\ell$ relations, compared to earlier chapters.

- Let $\alpha(\vec{k}), \beta(\vec{k}), \gamma(\vec{k})$ be the Euler angles for rotating the crd. system from the common crd. system to the $\hat{z}=\hat{k}$ crd. system. Then

$$a^T_{\vec{k}\ell m} = \sum_{m'} D^\ell_{mm'}(\alpha,\beta,\gamma)\,\tilde{a}^T_{\vec{k}\ell m'} \equiv \sum_{m'} D^\ell_{mm'}(\vec{k})\,\tilde{a}^T_{\vec{k}\ell m'} \tag{6}$$

Here $\beta = \vartheta$ is the angle between $\vec{k}$ and the z-axis of the common crd. system.

- From (6) & (5) we get

$$\underline{a^T_{\vec{k}\ell m} = i^\ell \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m'} D^\ell_{mm'}(\vec{k})\,\Theta^{m'}_{\ell\vec{k}}(\eta_0)} \tag{7}$$

which takes care of both issues 1) and 2).

- For polarization, we must also consider how $\hat{n} = \hat{n}_{phot}$ vs. $\hat{n} = \hat{n}_{obs}$ affects the definition of the Stokes parameters Q and U. In Chapter Y we said that its analysis applies equally well to both cases. But we need to choose which one we use. And now that we switch from one to the other, there's an effect.

- Q and U are defined with respect to a right-handed crd. system $(\hat{\vartheta}, \hat{\varphi}, \hat{n})$, where $\hat{\vartheta}$ and $\hat{\varphi}$ are the unit vectors in the direction of increasing ϑ and φ, on the unit sphere at $\hat{n}$. For $\hat{n} \to -\hat{n}$, $\hat{\varphi}$ changes direction but $\hat{\vartheta}$ is unchanged.

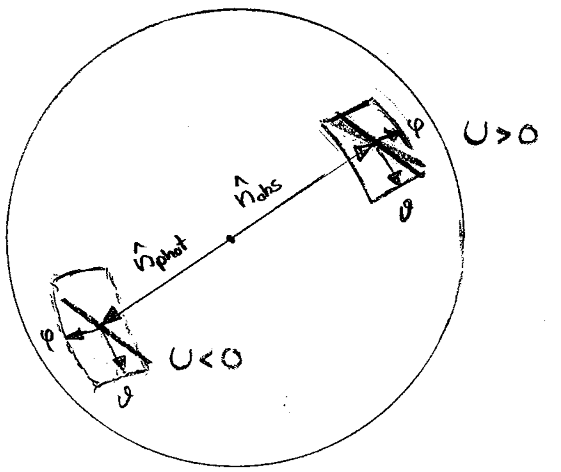

Fig. Polarization which has U>0 defined wrt $(\hat{\vartheta}_{obs}, \hat{\varphi}_{obs})$ has U<0 defined wrt $(\hat{\vartheta}_{phot}, \hat{\varphi}_{phot})$.

This changes the sign of U between the two definitions; whereas Q is unchanged.:

$$(Q \pm iU)_{obs}(\hat{n}) = (Q \mp iU)_{phot}(-\hat{n}) \tag{8}$$

- We can now do for polarization what we just did for temperature anisotropy:

$$(Q \pm iU)_{obs}(\hat{n}) \overset{(Y5.9)}{=} -\sum (a^E_{\ell m} \pm i a^B_{\ell m})\, {}_{\pm 2}Y_\ell^m(\hat{n})$$
$$= (Q \mp iU)_{phot}(\eta_0, \vec{x}_{obs}, -\hat{n}) = \sum_{\vec{k}} (Q \mp iU)^{phot}_{\vec{k}}(\eta_0, -\hat{n})\, \overbrace{e^{i\vec{k}\cdot\vec{x}_{obs}}}^{=1 \text{ (choose } \vec{x}_{obs}=0)} \tag{9}$$

and the observed multipoles are

$$a_{\pm 2,\ell m} \overset{(Y5.6)}{=} \int d\Omega\, {}_{\pm 2}Y_\ell^{m*}(\hat{n}) (Q \pm iU)_{obs}(\hat{n}) = \sum_{\vec{k}} \int d\Omega\, {}_{\pm 2}Y_\ell^{m*}(\hat{n}) (Q \mp iU)^{phot}_{\vec{k}}(\eta_0, -\hat{n})$$
$$\equiv \sum_{\vec{k}} a_{\pm 2, \vec{k}\ell m} \tag{10}$$

where $$a_{\pm2,\vec{k}Lm} \equiv \int d\Omega\, {}_{\pm2}Y_L^{m*}(\hat{n})(Q\mp iU)^{phot}_{\vec{k}}(y_0,-\hat{n}) = \int d\Omega \underbrace{{}_{\pm2}Y_L^{m*}(-\hat{n})}_{(-1)^L\,{}_{\mp2}Y_L^{m*}(-\hat{n})\ \ (Y5.13)}(Q\mp iU)^{phot}_{\vec{k}}(y_0,\hat{n})$$

$$= (-1)^L \int d\Omega\, {}_{\mp2}Y_L^{m*}(\hat{n})(Q\mp iU)^{phot}_{\vec{k}}(y_0,\hat{n}) \qquad (11)$$

is the $\vec{k}$ mode contribution to the polarization multipoles $a_{\pm2,Lm}$ in the common crd. system.

- According to Eq. (Y5.8) these can be combined into E and B multipoles as

$$a^E_{\vec{k}Lm} = -\tfrac{1}{2}(a_{2\vec{k}Lm} + a_{-2,\vec{k}Lm}) = -\tfrac{1}{2}(-1)^L \int d\Omega \left[{}_{-2}Y_L^{m*}(\hat{n})(Q-iU)^{phot}_{\vec{k}}(y_0,\hat{n}) + {}_{+2}Y_L^{m*}(\hat{n})(Q+iU)^{phot}_{\vec{k}}(y_0,\hat{n})\right]$$

$$a^B_{\vec{k}Lm} = \tfrac{i}{2}(a_{2,\vec{k}Lm} - a_{-2,\vec{k}Lm}) = \tfrac{i}{2}(-1)^L \int d\Omega \left[{}_{-2}Y_L^{m*}(\hat{n})(Q-iU)^{phot}_{\vec{k}}(y_0,\hat{n}) - {}_{+2}Y_L^{m*}(\hat{n})(Q+iU)^{phot}_{\vec{k}}(y_0,\hat{n})\right] \qquad (12)$$

- We can define $\tilde{a}_{\pm2,\vec{k}Lm}$, $\tilde{a}^E_{\vec{k}Lm}$, $\tilde{a}^B_{\vec{k}Lm}$ by the same eqs (11 and 12), but in the $\hat{z}=\hat{k}$ crd. system. In this crd. system we have earlier defined (Eq. C7.3)

$$E^m_{L\vec{k}}(y_0) \pm iB^m_{L\vec{k}}(y_0) = i^L\sqrt{\frac{2L+1}{4\pi}} \int d\Omega\, {}_{\pm2}Y_L^{m*}(\hat{n})(Q\pm iU)^{phot}_{\vec{k}}(y_0,\hat{n}) \qquad (13)$$

$$\Rightarrow \begin{cases} E^m_{L\vec{k}}(y_0) = \frac{1}{2} i^L\sqrt{\frac{2L+1}{4\pi}} \int d\Omega \left[{}_{+2}Y_L^{m*}(\hat{n})(Q+iU)^{phot}_{\vec{k}}(y_0,\hat{n}) + {}_{-2}Y_L^{m*}(\hat{n})(Q-iU)^{phot}_{\vec{k}}(y_0,\hat{n})\right] \\ \quad = -(-i)^L\sqrt{\frac{2L+1}{4\pi}}\,\tilde{a}^E_{\vec{k}Lm} \quad \text{(from Eq. 12)} \qquad (14) \\ B^m_{L\vec{k}}(y_0) = \frac{1}{2i} i^L\sqrt{\frac{2L+1}{4\pi}} \int d\Omega \left[{}_{+2}Y_L^{m*}(\hat{n})(Q+iU)^{phot}_{\vec{k}}(y_0,\hat{n}) - {}_{-2}Y_L^{m*}(\hat{n})(Q-iU)^{phot}_{\vec{k}}(y_0,\hat{n})\right] \\ \quad = (-i)^L\sqrt{\frac{2L+1}{4\pi}}\,\tilde{a}^B_{\vec{k}Lm} \qquad (15) \end{cases}$$

$$\Rightarrow \quad \tilde{a}^E_{\vec{k}Lm} = -i^L\sqrt{\frac{4\pi}{2L+1}}\,E^m_{L\vec{k}}(y_0) \quad \text{and} \quad \tilde{a}^B_{\vec{k}Lm} = i^L\sqrt{\frac{4\pi}{2L+1}}\,B^m_{L\vec{k}}(y_0) \qquad (16)$$

- We know from Section Y3 (Eq. Y3.12½) that the multipoles a^E_{Lm} and a^B_{Lm} transform just like ordinary a_{Lm} in a rotation of crd. system. Thus we have finally

$$\boxed{\begin{aligned} a^E_{\vec{k}Lm} &= -i^L\sqrt{\frac{4\pi}{2L+1}}\sum_{m'} D^L_{mm'}(\vec{k})\,E^{m'}_{L\vec{k}}(y_0) \\ a^B_{\vec{k}Lm} &= i^L\sqrt{\frac{4\pi}{2L+1}}\sum_{m'} D^L_{mm'}(\vec{k})\,B^{m'}_{L\vec{k}}(y_0) \end{aligned}} \qquad (17)$$