L5. Delta Function Approximation (good for illustration only)

- To get more insight into the dominant effects, we note that the visibility function is peaked around $\eta_* \equiv \eta_{dec}$, and approximate it by a delta function:

$$\boxed{\begin{aligned} g(\eta) &\simeq \delta(\eta-\eta_*) \quad (1) \\ \Rightarrow \quad e^{-\tau(\eta)} &\simeq \Theta(\eta-\eta_*) \quad (2) \end{aligned}}$$

(step function)

- That is, we approximate that the last scattering happened for each photon at exactly the same time.

- Here we completely ignore the effect at reionization. A straightforward generalization would be to use two delta functions (exercise)

$$g(\eta) \simeq e^{-\tau_{reion}} \cdot \delta(\eta-\eta_*) + \left(1-e^{-\tau_{reion}}\right) \delta(\eta-\eta_{reion}). \quad (3)$$

- With the approximation (1) & (2), Eq. (4.8) becomes

$$\boxed{\begin{aligned} \frac{1}{2l+1}\Theta_l^0(\eta_0) \simeq\ & \left[\Theta_0^0(\eta_*)+\phi(\eta_*)\right] j_l(k\eta_0-k\eta_*) + V_b^{(0)}(\eta_*)\, j_l'(k\eta_0-k\eta_*) \\ & + \underbrace{\int_{\eta_*}^{\eta_0} d\eta\, (\phi'+\psi')\, j_l(k\eta_0-k\eta)}_{\text{integrated Sachs-Wolfe effect}} + \underbrace{\frac{1}{10}\left[\Theta_2^0(\eta_*)-\sqrt{6}\,E_2^0(\eta_*)\right]}_{\text{local quadrupole effect}} j_l^{(20)}(k\eta_0-k\eta_*) \end{aligned}} \quad (4)$$

The effect of the local quadrupole is relatively small, and we drop it for the further discussion within the delta-function approximation. The result (4) is in the $k l m$ space, i.e. for multipoles at a single Fourier mode. The role of the radial functions $j_l^{(l'm)}$ is to project the plane-wave spatial dependence onto multipoles.

- In $\vec{x},\hat{n}$ space, Eq. (4), w/o the local quadrupole term, reads

$$\Theta(\eta_0,\vec{x}_{obs},\hat{n}) \simeq \underbrace{\Theta_0^0(\eta_*)+\Phi(\eta_*)}_{\text{effective temperature}} + \underbrace{\hat{n}\cdot\vec{v}_b(\eta_*)}_{\text{Doppler}} + \underbrace{\int_{\eta_*}^{\eta_0} d\eta\,(\Phi'+\Psi')}_{\text{ISW}} \tag{5}$$

which is Eq. (49) of Chapter 12 of my Cosmology II lecture notes; except that there we have approximated $\Psi \simeq \Phi$, and $\hat{n}$ has the opposite sign, since it refers to the observation direction, whereas here it refers to the photon direction.

- For vector ($m=\pm1$) and tensor ($m=\pm2$) modes, and for polarization, the delta function approximation gives, from (4.8–10) and (4.12–13),

$$\boxed{\begin{aligned}
\frac{1}{2l+1}\Theta_l^{\pm1}(\eta_0) &\simeq \left[V_b^{(\pm1)}(\eta_*)-B^{(\pm1)}(\eta_*)\right]j_l^{(11)}(k\eta_0-k\eta_*) + \int_{\eta_*}^{\eta_0} d\eta\,\frac{k}{\sqrt{3}}B^{(\pm1)}(\eta)\,j_l^{(21)}(k\eta_0-k\eta) \\
&\quad + \frac{1}{10}\left[\Theta_2^{\pm1}(\eta_*)-\sqrt{6}\,E_2^{\pm1}(\eta_*)\right]j_l^{(21)}(k\eta_0-k\eta_*) && (6)\\
\frac{1}{2l+1}\Theta_l^{\pm2}(\eta_0) &\simeq \frac{1}{10}\left[\Theta_2^{\pm2}(\eta_*)-\sqrt{6}\,E_2^{\pm2}(\eta_*)\right]j_l^{(22)}(k\eta_0-k\eta_*) - \int_{\eta_*}^{\eta_0} d\eta\,h^{(\pm2)\prime}(\eta)\,j_l^{(20)}(k\eta_0-k\eta) && (7)\\
\frac{1}{2l+1}E_l^{m}(\eta_0) &\simeq \frac{-\sqrt{6}}{10}\left[\Theta_2^{m}(\eta_*)-\sqrt{6}\,E_2^{m}(\eta_*)\right]\epsilon_l^{m}(k\eta_0-k\eta_*) && (8)\\
\frac{1}{2l+1}B_l^{m}(\eta_0) &\simeq \frac{-\sqrt{6}}{10}\left[\Theta_2^{m}(\eta_*)-\sqrt{6}\,E_2^{m}(\eta_*)\right]\beta_l^{m}(k\eta_0-k\eta_*) && (9)
\end{aligned}}$$

- We note that the temperature anisotropy $\Theta_l^m(\eta_0)$ gets a contribution both from the scattering surface and from the metric perturbations along the line of sight.
The <u>polarization</u>, $E_l^m(\eta_0)$, $B_l^m(\eta_0)$, on the other hand, <u>probes only the scattering surface</u>.
Hence CMB polarization observations have the power to separate scattering-surface effects from line-of-sight effects; that are mixed in temperature anisotropy observations. (But note that reionization introduces another "scattering surface", that we have ignored here; so in polarization observations the effects from these two "scattering surfaces" are mixed. Since the reionization surface is closer by, its effects appear at larger angular scales, i.e., at lower multipoles.)