L4. Optical Depth and Visibility Function

- From (2.22) and (1.2) we have for the present-day temperature anisotropy

$$\Theta_l^0(\eta_0) = (2l+1)\int_0^{\eta_0} d\eta\, e^{-\tau}\Big[(an_e\sigma_T\Theta_0^0+\psi')j_l(k\eta_0-k\eta) + (an_e\sigma_T v_b^{(0)}+k\phi)j_l'(k\eta_0-k\eta) + an_e\sigma_T P^{(0)} j_l^{(20)}(k\eta_0-k\eta)\Big] \tag{1}$$

$$\Theta_l^{\pm1}(\eta_0) = (2l+1)\int_0^{\eta_0} d\eta\, e^{-\tau}\Big[(an_e\sigma_T\cdot v_b^{(\pm1)} + B^{(\pm1)'})j_l^{(11)}(k\eta_0-k\eta) + an_e\sigma_T\cdot P^{(\pm1)} j_l^{(21)}(k\eta_0-k\eta)\Big] \tag{2}$$

$$\Theta_l^{\pm2}(\eta_0) = (2l+1)\int_0^{\eta_0} d\eta\, e^{-\tau}\left(an_e\sigma_T\cdot P^{(\pm2)} - h^{(\pm2)'}\right) j_l^{(22)}(k\eta_0-k\eta) \tag{3}$$

We note there are two kinds of source terms:

1) Those that are first generated by Thomson scattering, and then damped by it. These appear in the line-of-sight integral with a factor $an_e\sigma_T e^{-\tau}$

2) Those (the metric perturbations) whose effect is just damped by Thomson scattering. These appear with a factor $e^{-\tau}$

The polarization line-of-sight integrals (2.45) contain only the first kind.

- $an_e\sigma_T$ is the probability for a photon to scatter per conformal time $d\eta$

$\tau(\eta) \equiv \int_\eta^{\eta_0} an_e\sigma_T\, d\eta$, the optical depth, is the expectation value for the number of scatterings per photon between η and the present time η_0. (4)

$e^{-\tau}$ is the probability for a photon NOT to scatter between η and η_0,
i.e., the probability that it last scattered before η
i.e., the fraction of CMB photons we receive unscattered from η

$1-e^{-\tau}$ is the probability for a photon to scatter at least once between η and η_0

- For $\tau \ll 1$ we can neglect multiple scatterings for the same photon between η and η_0,

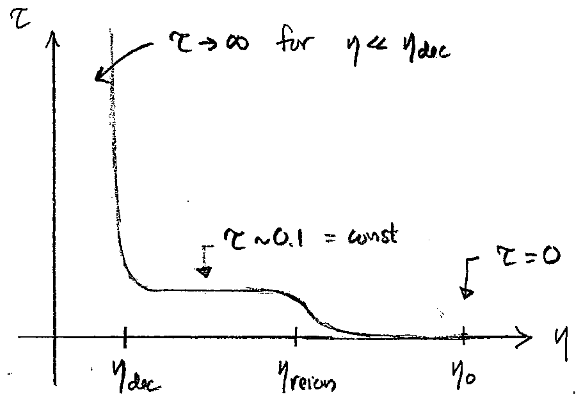
and $1-e^{-\tau} \simeq \tau \simeq$ the probability for a photon to scatter between η and η_0
= the fraction of the CMB photons that we see, that have scattered after η

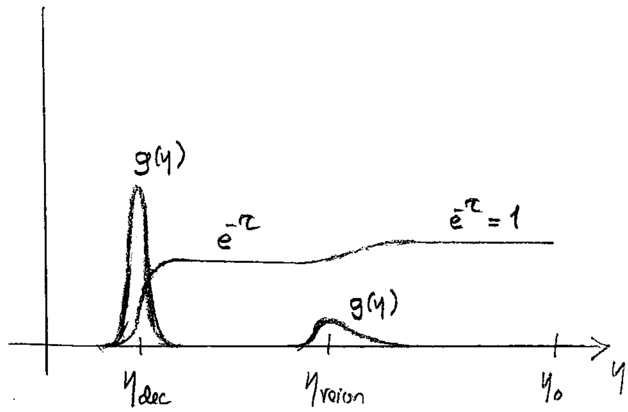
- From (4), $a n_e \sigma_T = -\tau'$. (5) We define the <u>visibility function</u>

$$\boxed{g(\eta) \equiv \frac{d}{d\eta}\left(e^{-\tau}\right) = -\tau' e^{-\tau} = a n_e \sigma_T e^{-\tau}.} \qquad (6)$$

For a given CMB photon, $g(\eta)d\eta$ gives the probability that it <u>last</u> scattered during the time interval $(\eta, \eta+d\eta)$. Thus the visibility function $g(\eta)$ gives the distribution (*) in (conformal) time η, and in (comoving) distance $r = \eta_0 - \eta$, of where we are looking when we are looking at the CMB.

- For most CMB photons, the last scattering occurs near the photon decoupling time η_{dec}, the so-called last scattering surface. However, some CMB photons scattered during or after the reionization at η_{reion}. See Figure.





- In the context of cosmological parameter estimation, the parameter τ refers to the constant value $\tau(\eta)$ has for $\eta_{dec} \ll \eta \ll \eta_{reion}$, and is often referred to as the optical depth to the last scattering surface, or the optical depth due to reionization.

- Thus the sources in (1)–(3), which contains the factor $g(\eta) = a n_e \sigma_T e^{-\tau}$ are localized at the "scattering surfaces"; mainly at the "last scattering surface" at $\eta \sim \eta_{dec}$, but also some at the reionization epoch, $\eta \sim \eta_{reion}$.

  Whereas the metric sources, which only have the factor $e^{-\tau}$, are distributed all along the line-of-sight, back to the photon decoupling. Their effect from η gets reduced by the fraction of photons that scattered after η.

*) Note that $\int_0^{\eta_0} g(\eta)\,d\eta = e^{-\tau(\eta_0)} - e^{-\tau(0)} = e^0 - e^{-\infty} = 1 - 0 = 1.$ (7)

- The metric source term $k\phi$ in (1) comes from the gradient of the Bardeen potential ϕ. Both the time derivative ϕ' and the gradient $\nabla\phi = i\vec{k}\phi$ contribute to the evolution along the line of sight (the path of the CMB photons to us after their last scattering). It turns out to be helpful to redistribute the effect of ϕ in (1) by partial integration; so that the gradient $k\phi$ gets replaced by the time derivative ϕ' along the photon path and the total difference in ϕ between last scattering and observation. The ϕ at the observing site is independent of the direction we are looking at, and thus contributes only to the monopole.

$$\frac{d}{dy}\left[e^{-\tau}\phi\, j_l(ky_0-ky)\right] = -\tau' e^{-\tau}\phi\, j_l + e^{-\tau}\phi' j_l - e^{-\tau}\phi k j_l'$$

$$\Rightarrow \int_0^{y_0} dy\, e^{-\tau} k\phi j_l' = \underbrace{-\Big/_{0}^{y_0} e^{-\tau}\phi j_l(ky_0-ky)}_{=\ \phi(y_0)\delta_{l0}} + \int_0^{y_0}\left[g(y)\phi j_l + e^{-\tau(y)}\phi' j_l\right]dy$$

since $e^{-\tau(0)} = e^{-\infty} = 0$, $e^{-\tau(y_0)} = e^0 = 1$,

and $j_l(ky_0-ky_0) = j_l(0) = \delta_{l0}$ (3.4)

- Similarly, we can redistribute the metric shift $B^{(\pm1)\prime}$ in (2), using $j_l^{(11)\prime} = \frac{1}{\sqrt{3}} j_l^{(21)}$ from Eq. (3.10). (Exercise).

- Thus Eqs. (1)-(3) can be rewritten as (no partial integration done for (3))

$$\Theta_l^0(y_0) = (2l+1)\int_0^{y_0} dy \Big\{ g(y)\Big[\underline{(\Theta_0^0+\phi) j_l(ky_0-ky) + V_b^{(0)} j_l^{(10)}(ky_0-ky) + P^{(0)} j_l^{(20)}(ky_0-ky)}\Big] + \underline{e^{-\tau(y)}(\phi'+\psi') j_l(ky_0-ky)}\Big\} - \underline{\phi(y_0)\delta_{l0}} \qquad (8)$$

$$\Theta_l^{\pm1}(y_0) = (2l+1)\int_0^{y_0} dy \Big\{ g(y)\Big[\underline{(V_b^{(\pm1)} - B^{(\pm1)}) j_l^{(11)}(ky_0-ky) + P^{(\pm1)} j_l^{(21)}(ky_0-ky)}\Big] + e^{-\tau(y)}\cdot\underline{\frac{1}{\sqrt{3}} k B^{(\pm1)} j_l^{(21)}(ky_0-ky)}\Big\} + \underline{B^{(\pm1)}(y_0)\delta_{l1}} \qquad (9)$$

$$\Theta_l^{\pm2}(y_0) = (2l+1)\int_0^{y_0} dy \Big\{ g(y)\cdot\underline{P^{(\pm2)} j_l^{(22)}(ky_0-ky)} - e^{-\tau(y)}\, \underline{h^{(\pm2)\prime} j_l^{(22)}(ky_0-ky)}\Big\} \qquad (10)$$

- In practice, we can not separate the monopole contribution $-\phi(\eta_0)\delta_{l0}$ due to gravitational potential at observing site from the background CMB temperature T_0. Nor can we separate the dipole contribution $B^{(\pm 1)}(\eta_0)\delta_{l1}$ due to metric shift vector from the Doppler effect due to observer (Solar System) motion. Therefore only $l \geq 2$ multipoles are usually considered, when CMB anisotropy observations are applied to cosmology.

- A side effect of the partial integration of (1) → (8) was to convert $j_l^{(10)} = j_l'$ to $j_l^{(00)} = j_l$. Additional partial integrations could be applied to (8) to convert also the remaining $j_l^{(10)}$ and $j_l^{(20)}$ to j_l, at the cost of introducing gradients and time derivatives to the other factors (and boundary terms for $l=0$ and $l=1$). This was done in the 2004 CMB Physics course; but here we shall not do it, and keep instead the source terms otherwise simpler, as we have invested in understanding the more general radial functions $j_l^{(l'm)}$.

- The combination $\underline{\Theta_0^0(\eta) + \phi(\eta) = \frac{1}{4}\delta_\gamma(\eta) + \phi(\eta)}$ is called the <u>effective temperature</u>. (11)
  It adds to the monopole of the local temperature perturbation the gravitational blue/redshift that would result from the photons falling down / climbing up from the local gravitational potential perturbation. The other terms in (8) are the Doppler effect, the effect of the local quadrupole anisotropy and polarization, and the integrated Sachs-Wolfe (ISW) effect from the time evolution of the potentials ϕ and ψ along the line of sight.

- In like manner, (2.45) can be written directly (no partial integration needed) as

$$\boxed{\begin{aligned} E_l^m(\eta_0) &= -\sqrt{6}\,(2l+1)\int_0^{\eta_0} d\eta\; g(\eta)\cdot \underline{P^{(m)}\cdot \epsilon_l^m(k\eta_0 - k\eta)} && (12) \\ B_l^m(\eta_0) &= -\sqrt{6}\,(2l+1)\int_0^{\eta_0} d\eta\; g(\eta)\cdot \underline{P^{(m)}\cdot \beta_l^m(k\eta_0 - k\eta)} && (13) \end{aligned}}$$