L3. Radial Functions

- The radial functions $j_j^{(\ell' m)}(x)$ and $_{\pm 2}j_j^{(\ell' m)}(x)$ determine the free streaming of multipoles according to Eqs. (2.15) and (2.38). For the line-of-sight integrals (2.22) and (2.45) we need the $j_j^{(\ell' m)}$ only up to $\ell' = 2$ (and $m = 0, \ldots, \pm \ell'$), and $_{\pm 2}j_j^{(2m)} = \varepsilon_j^m \pm i\beta_j^m$ for $m = 0, \pm 1, \pm 2$.

- The radial functions $j_j^{(\ell' m)}$ are given by the Clebsch-Gordan series as (2.14)

$$\underline{j_j^{(\ell' m)} \equiv \sum_\ell (-i)^{\ell+\ell'-j} \left(\frac{2\ell+1}{2j+1}\right) \langle \ell 0 \ell' m | j m \rangle \langle \ell 0 \ell' 0 | j 0 \rangle \, j_\ell} \tag{1}$$

as linear combinations of spherical Bessel functions.

- We shall need some properties of the spherical Bessel functions $j_\ell(x)$:

Recursion formulae:

($'$ $\equiv$ derivative)

$$\boxed{\begin{aligned} (2\ell+1)\, j_\ell' &= \ell \cdot j_{\ell-1} - (\ell+1) \cdot j_{\ell+1} \qquad (2) \\ (2\ell+1)\, \frac{j_\ell(x)}{x} &= j_{\ell-1}(x) + j_{\ell+1}(x) \qquad (3) \end{aligned}}$$

Value at zero:

$$\underline{j_\ell(0) = \delta_{\ell 0}} \tag{4}$$

- For $(\ell' m) = (00)$, we simply have $j_j^{(00)} = j_j$ since $\langle \ell 0 0 0 | j 0 \rangle = \delta_{\ell j}$.

In general, the $\langle \ell 0 \ell' m | j m \rangle$ can be nonzero only for $\ell = |j - \ell'|, \ldots, j + \ell'$.

Moreover, since $\langle \ell 0 \ell' 0 | j 0 \rangle = 0$ for $\ell + \ell' - j$ odd, <u>(1) turns out to be real,</u> and the $\ell' = 1$ case has only 2 terms: $\ell = j-1, j+1$ and the $\ell' = 2$ case has only 3 terms: $\ell = j-2, 0, j+2$ in the series (1).

- Using the values for $\langle \ell 0 \ell' m | j m \rangle$ given by VMK Tables 8.2 and 8.4 and then the recursion formulae (2) and (3) we get the following results for $j_j^{(\ell' m)}$ (next page).

- The radial functions $j_L^{(l'm)}(x)$

$$\boxed{\begin{aligned}
j_L^{(00)} &= j_L \\
j_L^{(10)} &= \frac{L}{2L+1} j_{L-1} - \frac{L+1}{2L+1} j_{L+1} \\
j_L^{(20)} &= \frac{3}{2}\frac{L(L-1)}{(2L+1)(2L-1)} j_{L-2} - \frac{L(L+1)}{(2L+3)(2L-1)} j_L + \frac{3}{2}\frac{(L+2)(L+1)}{(2L+3)(2L+1)} j_{L+2} \\
j_L^{(11)} &= \frac{1}{2L+1}\sqrt{\frac{L(L+1)}{2}}\left(j_{L-1} + j_{L+1}\right) \\
j_L^{(21)} &= \sqrt{\frac{3}{2}L(L+1)}\left[\frac{L-1}{(2L+1)(2L-1)} j_{L-2} - \frac{1}{(2L-1)(2L+3)} j_L - \frac{L+2}{(2L+1)(2L+3)} j_{L+2}\right] \\
j_L^{(22)} &= \sqrt{\frac{3}{8}(L-1)L(L+1)(L+2)}\left[\frac{1}{(2L+1)(2L-1)} j_{L-2} + \frac{2}{(2L-1)(2L+3)} j_L + \frac{1}{(2L+1)(2L+3)} j_{L+2}\right]
\end{aligned}} \qquad (5)$$

$$\underline{j_L^{(l',-m)} = j_L^{(l'm)}} \qquad (6)$$

- Using the recursion formulae (2) & (3) we can put these into a form where only the same j_L appears in each term:

$$\boxed{\begin{aligned}
j_L^{(00)} &= j_L & j_L^{(11)}(x) &= \sqrt{\frac{L(L+1)}{2}}\frac{j_L(x)}{x} \\
j_L^{(10)} &= j_L' & j_L^{(21)}(x) &= \sqrt{\frac{3}{2}L(L+1)}\left(\frac{j_L(x)}{x}\right)' \\
j_L^{(20)} &= \frac{3}{2}j_L'' + \frac{1}{2}j_L & j_L^{(22)}(x) &= \sqrt{\frac{3}{8}(L-1)L(L+1)(L+2)}\frac{j_L(x)}{x^2}
\end{aligned}} \qquad (7)$$

- To show by substitution that (2.22) satisfies (2.1a) we need the recursion relation

derivative

$$\underline{(2L+1)\, j_L^{(l'm)\prime} = \sqrt{L^2-m^2}\, j_{L-1}^{(l'm)} - \sqrt{(L+1)^2-m^2}\, j_{L+1}^{(l'm)}} \qquad (8)$$

(I have verified this only for (00), (10), (20), (11)) and the values at zero

$$\underline{j_L^{(l'm)}(0) = \frac{1}{2L+1}\delta_{Ll'}} \qquad (9)$$

- Note also that

$$j_L^{(00)\prime} = j_L^{(10)}$$

$$j_L^{(10)\prime} = \frac{2}{3}j_L^{(20)} - \frac{1}{3}j_L^{(00)} \qquad j_L^{(11)\prime} = \frac{1}{\sqrt{3}}j_L^{(21)} \qquad (10)$$

- The polarization radial functions ${}_{\pm2}j_j^{(\ell' m)}$ are given by (36) as

$$\underline{{}_{\pm 2}j_j^{(2m)} \equiv \sum_{\ell} (-i)^{\ell+2-j} \left(\frac{2\ell+1}{2j+1}\right) \langle \ell 0 2m | jm \rangle \langle \ell 0; 2, \mp 2 | j, \mp 2 \rangle \, j_\ell} \qquad (11)$$

- The complex phase of the terms comes from $(-i)^{\ell+2-j}$ $\Rightarrow$ the $\ell = j-1, j+1$ terms are imaginary and the others are real. Examination of VMK Table 8.4 reveals that

$$\langle j\pm1, 0; 2, +2 | j, +2 \rangle = - \langle j\pm1, 0; 2, -2 | j, -2 \rangle \quad \Rightarrow \quad \underline{{}_{-2}j_j^{(2m)} = {}_{+2}j_j^{(2m)*}} \qquad (12)$$

$\therefore$ Thus we can write $\underline{{}_{\pm2}j_j^{(2m)} = \epsilon_j^m \pm i\beta_j^m}$ where ϵ_j^m, β_j^m are real. $\square$ (13)

- From VMK Table 8.4 we also see that $\langle \ell 0; 2, -m | j, -m \rangle = (-1)^{\ell-j} \langle \ell 0; 2m | j, m \rangle$, meaning that the imaginary terms (i.e. the β_j^m) get the opposite sign for $-m$. .

$$\Rightarrow \quad \underline{{}_{2}j_j^{(2,-m)} = {}_{2}j_j^{(2m)*}} \quad \Rightarrow \quad \boxed{\begin{aligned} \epsilon_j^{-m} &= \epsilon_j^m \\ \beta_j^{-m} &= -\beta_j^m \end{aligned}} \qquad (14)$$

Combining (12) & (14), we have $\underline{{}_{2}j_j^{(2,-m)} = {}_{2}j_j^{(2m)*} = {}_{-2}j_j^{(2m)} = {}_{-2}j_j^{(2,-m)*}}$ (15)

We also see from (14) that $\underline{\beta_j^0 = 0}$. (16)

- Comparing (11) to (1), we note that ${}_{\pm2}j_j^{(20)} = \underline{\epsilon_j^0 = j_j^{(2,\mp2)} = j_j^{(22)}}$ (17)

- Using VMK Table 3.4 we get from (11) & (13) that

$$
\boxed{
\begin{aligned}
\mathcal{E}_l^0 &= j_l^{(22)} \\
\mathcal{E}_l^1 &= \tfrac{1}{2}\sqrt{(l-1)(l+2)}\left[\frac{l+1}{(2l+1)(2l-1)}\, j_{l-2} + \frac{3}{(2l-1)(2l+3)}\, j_l - \frac{l}{(2l+1)(2l+3)}\, j_{l+2}\right] \\
\mathcal{E}_l^2 &= \tfrac{1}{4}\left[\frac{(l+1)(l+2)}{(2l-1)(2l+1)}\, j_{l-2} - 6\,\frac{(l-1)(l+2)}{(2l-1)(2l+3)}\, j_l + \frac{(l-1)l}{(2l+1)(2l+3)}\, j_{l+2}\right] \\
\beta_l^0 &= 0 \\
\beta_l^1 &= \frac{\sqrt{(l-1)(l+2)}}{2(2l+1)}\left[j_{l-1} + j_{l+1}\right] \\
\beta_l^2 &= \frac{1}{2(2l+1)}\left[(l+2)j_{l-1} - (l-1)j_{l+1}\right]
\end{aligned}
}
\tag{18}
$$

- Using the recursion relations (2) & (3) we can put these into the form

$$
\boxed{
\begin{aligned}
\mathcal{E}_l^0(x) &= \sqrt{\tfrac{3}{8}(l-1)l(l+1)(l+2)}\,\frac{j_l(x)}{x^2} & \beta_l^0(x) &= 0 \\
\mathcal{E}_l^1(x) &= \tfrac{1}{2}\sqrt{(l-1)(l+2)}\left[\frac{j_l(x)}{x^2} + \frac{j_l'(x)}{x}\right] & \beta_l^1(x) &= \tfrac{1}{2}\sqrt{(l-1)(l+2)}\,\frac{j_l(x)}{x} \\
\mathcal{E}_l^2(x) &= \tfrac{1}{4}\left[-j_l(x) + j_l''(x) + 2\frac{j_l(x)}{x^2} + 4\frac{j_l'(x)}{x}\right] & \beta_l^2(x) &= \tfrac{1}{2}\left[j_l'(x) + 2\frac{j_l(x)}{x}\right]
\end{aligned}
}
\tag{19}
$$

$C^{c\gamma}_{a\alpha b\beta} \equiv \langle a\alpha\, b\beta | c\gamma \rangle$

Table 8.2.

$C^{c\gamma}_{a\alpha 1 \beta}$

c	$\beta = 1$	$\beta = 0$	$\beta = -1$
$a+1$	$\left[\frac{(c+\gamma-1)(c+\gamma)}{(2c-1)\,2c}\right]^{1/2}$	$\left[\frac{(c+\gamma)(c-\gamma)}{(2c-1)\,c}\right]^{1/2}$	$\left[\frac{(c-\gamma-1)(c-\gamma)}{(2c-1)\,2c}\right]^{1/2}$
a	$-\left[\frac{(c+\gamma)(c-\gamma+1)}{2c\,(c+1)}\right]^{1/2}$	$\frac{\gamma}{[c\,(c+1)]^{1/2}}$	$\left[\frac{(c+\gamma+1)(c-\gamma)}{2c\,(c+1)}\right]^{1/2}$
$a-1$	$\left[\frac{(c-\gamma+1)(c-\gamma+2)}{(2c+2)(2c+3)}\right]^{1/2}$	$-\left[\frac{(c+\gamma+1)(c-\gamma+1)}{(c+1)(2c+3)}\right]^{1/2}$	$\left[\frac{(c+\gamma+2)(c+\gamma+1)}{(2c+2)(2c+3)}\right]^{1/2}$

Table 8.4

$C^{c\gamma}_{a\alpha 2 \beta}$

c	$\beta = 2$	$\beta = 1$
$a+2$	$\left[\frac{(c+\gamma-3)(c+\gamma-2)(c+\gamma-1)(c+\gamma)}{(2c-3)(2c-2)(2c-1)\,2c}\right]^{1/2}$	$\left[\frac{(c+\gamma-2)(c+\gamma-1)(c+\gamma)(c-\gamma)}{(2c-3)(c-1)(2c-1)\,c}\right]^{1/2}$
$a+1$	$-\left[\frac{(c+\gamma-2)(c+\gamma-1)(c+\gamma)(c-\gamma+1)}{(2c-2)(2c-1)\,c\,(c+1)}\right]^{1/2}$	$-(c-2\gamma+1)\left[\frac{(c+\gamma-1)(c+\gamma)}{(2c-2)(2c-1)\,c\,(c+1)}\right]^{1/2}$
a	$\left[\frac{3\,(c+\gamma-1)(c+\gamma)(c-\gamma+1)(c-\gamma+2)}{2\,(2c-1)\,c\,(c+1)(2c+3)}\right]^{1/2}$	$(1-2\gamma)\left[\frac{3\,(c+\gamma)(c-\gamma+1)}{2\,(2c-1)\,c\,(c+1)(2c+3)}\right]^{1/2}$
$a-1$	$-\left[\frac{(c+\gamma)(c-\gamma+1)(c-\gamma+2)(c-\gamma+3)}{c\,(c+1)(2c+3)(2c+4)}\right]^{1/2}$	$(c+2\gamma)\left[\frac{(c-\gamma+1)(c-\gamma+2)}{c\,(c+1)(2c+3)(2c+4)}\right]^{1/2}$
$a-2$	$\left[\frac{(c-\gamma+1)(c-\gamma+2)(c-\gamma+3)(c-\gamma+4)}{(2c+2)(2c+3)(2c+4)(2c+5)}\right]^{1/2}$	$-\left[\frac{(c+\gamma+1)(c-\gamma+1)(c-\gamma+2)(c-\gamma+3)}{(c+1)(2c+3)(c+2)(2c+5)}\right]^{1/2}$

Table 8.4. (*Cont.*)

c	$\beta = 0$	$\beta = -1$	$\beta = -2$
$a+2$	$\left[\frac{3\,(c+\gamma-1)(c+\gamma)(c-\gamma-1)(c-\gamma)}{(2c-3)(2c-2)(2c-1)\,c}\right]^{1/2}$	$\left[\frac{(c+\gamma)(c-\gamma-2)(c-\gamma-1)(c-\gamma)}{(2c-3)(c-1)(2c-1)\,c}\right]^{1/2}$	$\left[\frac{(c-\gamma-3)(c-\gamma-2)(c-\gamma-1)(c-\gamma)}{(2c-3)(2c-2)(2c-1)\,2c}\right]^{1/2}$
$a+1$	$\gamma\left[\frac{3\,(c+\gamma)(c-\gamma)}{(c-1)(2c-1)\,c\,(c+1)}\right]^{1/2}$	$(c+2\gamma+1)\left[\frac{(c-\gamma-1)(c-\gamma)}{(2c-2)(2c-1)\,c\,(c+1)}\right]^{1/2}$	$\left[\frac{(c+\gamma+1)(c-\gamma-2)(c-\gamma-1)(c-\gamma)}{(2c-2)(2c-1)\,c\,(c+1)}\right]^{1/2}$
a	$\frac{3\gamma^2-c\,(c+1)}{[(2c-1)\,c\,(c+1)(2c+3)]^{1/2}}$	$(2\gamma+1)\left[\frac{3\,(c+\gamma+1)(c-\gamma)}{2\,(2c-1)\,c\,(c+1)(2c+3)}\right]^{1/2}$	$\left[\frac{3\,(c+\gamma+1)(c+\gamma+2)(c-\gamma-1)(c-\gamma)}{2\,(2c-1)\,c\,(c+1)(2c+3)}\right]^{1/2}$
$a-1$	$-\gamma\left[\frac{3\,(c+\gamma+1)(c-\gamma+1)}{c\,(c+1)(2c+3)(c+2)}\right]^{1/2}$	$-(c-2\gamma)\left[\frac{(c+\gamma+1)(c+\gamma+2)}{c\,(c+1)(2c+3)(2c+4)}\right]^{1/2}$	$\left[\frac{(c+\gamma+1)(c+\gamma+2)(c+\gamma+3)(c-\gamma)}{c\,(c+1)(2c+3)(2c+4)}\right]^{1/2}$
$a-2$	$\left[\frac{3\,(c+\gamma+1)(c+\gamma+2)(c-\gamma+1)(c-\gamma+2)}{(c+1)(2c+3)(2c+4)(2c+5)}\right]^{1/2}$	$-\left[\frac{(c+\gamma+1)(c+\gamma+2)(c+\gamma+3)(c-\gamma+1)}{(c+1)(2c+3)(c+2)(2c+5)}\right]^{1/2}$	$\left[\frac{(c+\gamma+1)(c+\gamma+2)(c+\gamma+3)(c+\gamma+4)}{(2c+2)(2c+3)(2c+4)(2c+5)}\right]^{1/2}$

$j_\ell^{(10)}(x) = j'_\ell(x),$

$j''_\ell(x) + j_\ell(x)],$

$\overline{\ell(\ell+1)}\, j_\ell(x)$
$\overline{2 x}\,,$

$\overline{(\ell+1)}\left(\frac{j_\ell(x)}{x}\right)',$
2

$\overline{3\ (\ell+2)!}\, j_\ell(x)$
$3\ (\ell-2)!\ \ x^2$ (15)

ivatives with respect to the ar- $x = kr$. These modes are shown

functions with $m > 0$ (see Fig.

$\overline{4\pi(2\ell+1)}[\epsilon_\ell^{(m)}(kr)$

$)]_{\pm 2}Y_\ell^m(\hat{n}),$ (16)

$\overline{(\ell+2)!}\, j_\ell(x)$
$(\ell-2)!\ \ x^2\,,$

$\overline{\ell+2)}\left[\frac{j_\ell(x)}{x^2} + \frac{j'_\ell(x)}{x}\right],$

$''_\ell(x) + 2\frac{j_\ell(x)}{x^2} + 4\frac{j'_\ell(x)}{x}\Bigg],$ (17)

$= \ell, \ell \pm 2$ coupling and

$x) = 0,$

$\overline{-1)(\ell+2)}\frac{j_\ell(x)}{x},$

$'_\ell(x) + 2\frac{j_\ell(x)}{x}\Bigg],$ (18)

$' = \ell \pm 1$ coupling. The corre-

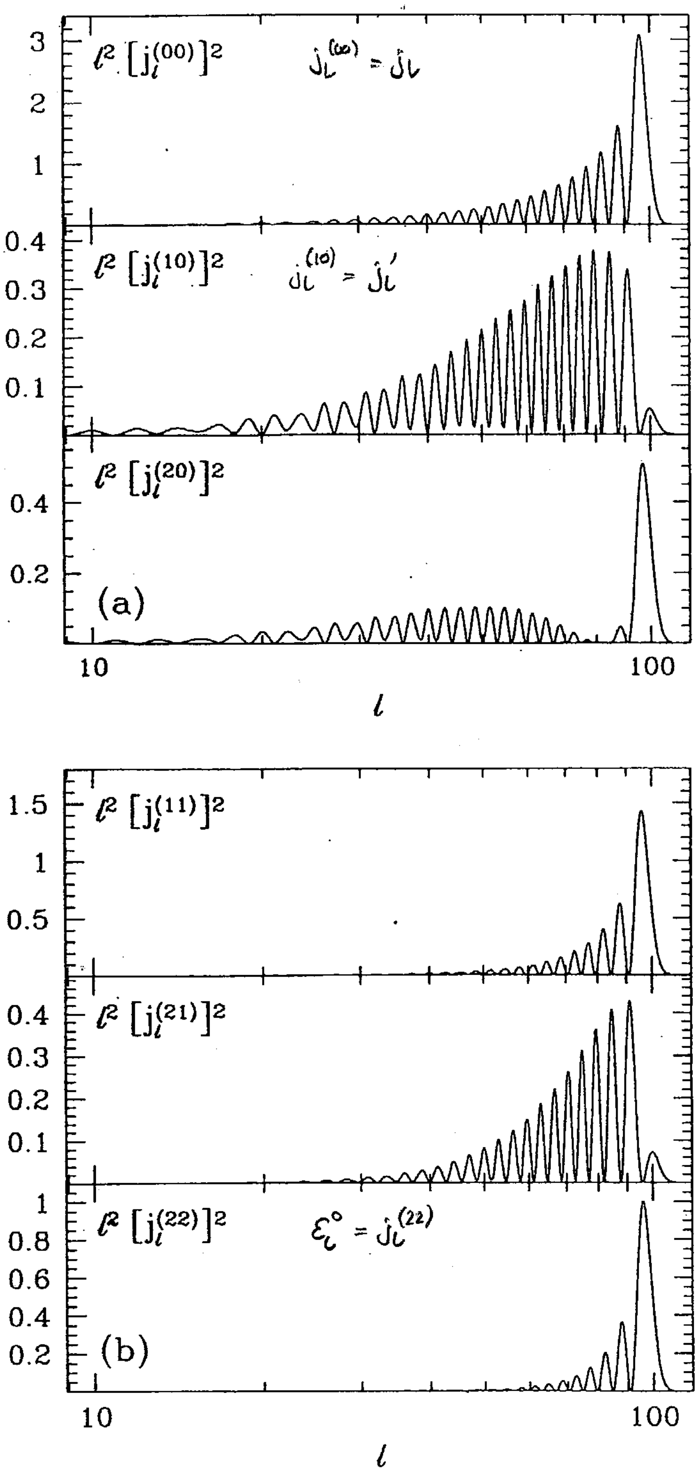

FIG. 3. Radial spin-0 (temperature) modes. The angular power in a plane wave (left panel, top) is modified due to the intrinsic angular structure of the source as discussed in the text. The left panel corresponds to the power in scalar ($m = 0$) monopole G_0^0, dipole G_1^0, and quadrupole G_2^0 sources (top to bottom); the right panel to that in vector ($m = 1$) dipole $G_1^{\pm 1}$ and quadrupole $G_2^{\pm 1}$ sources and a tensor ($m = 2$) quadrupole $G_2^{\pm 2}$ source (top to bottom). Note the differences in how sharply peaked the power is at $\ell \approx kr$ and how fast power falls as $\ell \ll kr$. The argument of the radial functions $kr = 100$ here.

to a range of angular scales from $\ell \approx kr$ at $\theta = \pi/2$ to larger

L3.7

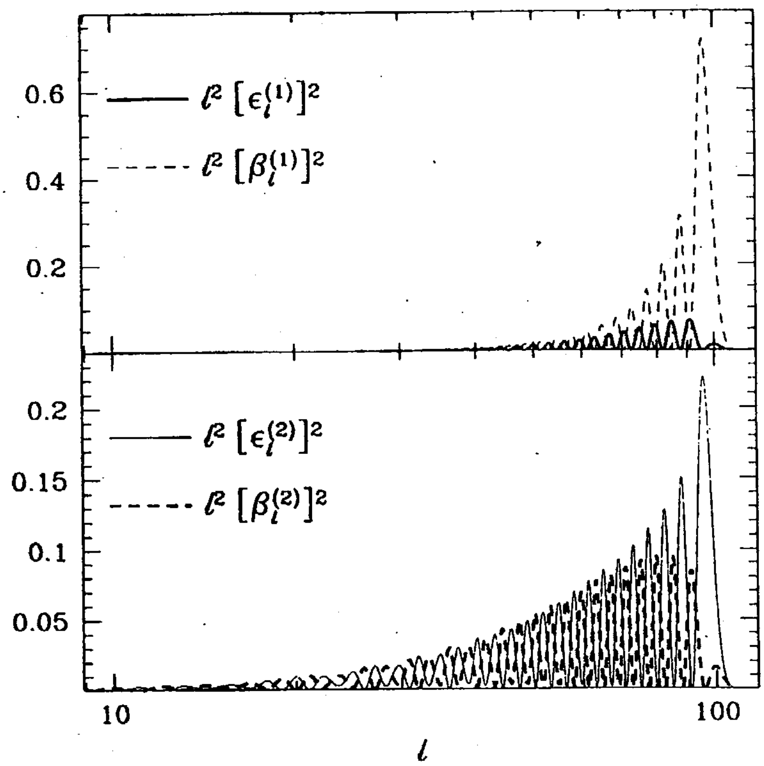

FIG. 4. Radial spin-2 (polarization) modes. Displayed is the angular power in a plane-wave spin-2 source. The top panel shows that vector ($m=1$, upper panel) sources are dominated by B-parity contributions, whereas tensor ($m=2$, lower panel) sources have comparable but less power in the B parity. Note that the power is strongly peaked at $\ell=kr$ for the B-parity vectors and E-parity tensors. The argument of the radial functions $kr=100$ here.

Thus factors of $\sin\theta$ in the intrinsic angular dependence suppress power at $\ell \ll kr$ ("aliasing suppression"), whereas factors of $\cos\theta$ suppress power at $\ell \approx kr$ ("projection suppression"). Let us consider first a $m=0$ dipole contribution $Y_1^0 \propto \cos\theta$ (see Fig. 2). The $\cos\theta$ dependence suppresses power in $j_\ell^{(10)}$ at the peak in the plane-wave spectrum $\ell \approx kr$ [compare Fig. 3(a) top and middle panels]. The remaining power is broadly distributed for $\ell \lesssim kr$. The same reasoning applies for Y_2^0 quadrupole sources which have an intrinsic angular dependence of $3\cos^2\theta - 1$. Now the minimum falls at $\theta = \cos^{-1}(1/\sqrt{3})$ causing the double peaked form of the power in $j_\ell^{(20)}$ shown in Fig. 3(a) (bottom panel). This series can be continued to higher G_ℓ^0 and such techniques have been used in the free streaming limit for temperature anisotropies [11].

Similarly, the structures of $j_\ell^{(11)}$, $j_\ell^{(21)}$, and $j_\ell^{(22)}$ are apparent from the intrinsic angular dependences of the G_1^1,

Secondly, even
wavelength sources
angle anisotropies a

This scaling puts a
can rise with ℓ tha
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that *decreases* with

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tions ϵ_ℓ and β_ℓ.
introduces a β cont
$m=\pm 1$, the β cor
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reach the asymptoti

$$\frac{\Sigma_\ell[\ell}{\Sigma_\ell[\ell}$$

for fixed $kr \gg 1$. Tl
the parity of the m
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does mix states of
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in the intrinsic ang

$${}_2G_2^m \mathbf{M}_+ + {}_{-2}G$$

Thus the addition
generates "magnet
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