L2. Line-of-Sight Integration

- We shall consider a single Fourier mode $\vec{k}$. Thus the brightness function and the linear polarization Stokes parameters in $\vec{x},\hat{n}$ space are

$$\Theta(\eta,\vec{x},\hat{n}) = \Theta(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}} \tag{1}$$

$$(Q+iU)(\eta,\vec{x},\hat{n}) = (Q+iU)(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}} \tag{2}$$

L2.1 Temperature Anisotropy

- Consider first just free streaming. Its effect in the ordinary $(\vec{x},\hat{n})$ space is very simple: From conformal time η_1 to η_2 each photon travels the comoving distance $r = \eta_2 - \eta_1$ in the direction $\hat{n}$ of its momentum. Since all photons with the same $\hat{n}$ travel together, the brightness function value $\Theta(\eta_1,\vec{x},\hat{n})$ is transported to $\Theta(\eta_2, \vec{x}+r\hat{n},\hat{n})$.

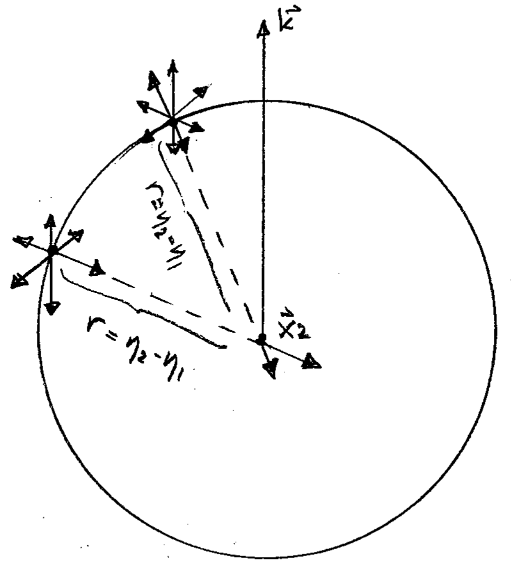

Fig. Free streaming of temperature anisotropy from time η_1 to η_2.

In other words, the brightness function at $\eta_2, \vec{x}_2$ can be reconstructed from its values at η_1 as (see Figure)

$$\Theta(\eta_2,\vec{x}_2,\hat{n}) = \Theta(\eta_1, \vec{x}_2 - r\hat{n}, \hat{n}) \tag{3}$$

- Consider now multipoles. Eq. (1) can be written as

$$\Theta(\eta,\vec{x},\hat{n}) = \sum_{\ell m} (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, \Theta_\ell^m(\eta)\, Y_\ell^m(\hat{n})\, e^{i\vec{k}\cdot\vec{x}} = \sum_{\ell m} \Theta_\ell^m(\eta)\, G_{\ell\vec{k}}^m(\vec{x},\hat{n}) \tag{4}$$

where the mode function $G_{\ell\vec{k}}^m$ is

$$G_{\ell\vec{k}}^m(\vec{x},\hat{n}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m(\hat{n})\, e^{i\vec{k}\cdot\vec{x}} \tag{5}$$

and we use the coordinate system (for Y_ℓ^m) where z-axis is in the $\vec{k}$ direction.

- Eq. (3) then becomes

$$\Theta(\eta_2,\vec{x}_2,\hat{n}) = \sum_{lm} \Theta_l^m(\eta_1)\, G_{l\vec{k}}^m(\vec{x}_2 - r\hat{n},\hat{n}) = e^{i\vec{k}\cdot\vec{x}_2} \overbrace{\sum_{lm} \Theta_l^m(\eta_1)\, G_{lk}^m(r,\hat{n})}^{\Theta(\eta_2,\hat{n})} \tag{6}$$

where we have defined the line-of-sight mode function $\qquad \vec{k}\cdot\vec{x} = -kr\hat{z}\cdot\hat{n} = -kr\cos\theta$

$$G_{lk}^m(r,\hat{n}) \equiv G_{l\vec{k}}^m(-r\hat{n},\hat{n}) = (-i)^l \sqrt{\frac{4\pi}{2l+1}}\, Y_l^m(\hat{n})\, e^{-ikr\hat{z}\cdot\hat{n}} \tag{7}$$

(that is, only these pairs $\vec{x},\hat{n}$ are considered, where $\vec{x}$ is in the $-\hat{n}$ direction).

- To relate $\Theta_l^m(\eta_2)$ to $\Theta_l^m(\eta_1)$ we just need to pick out the multipoles of Eq. (6). For that we need the spherical harmonic expansion of the plane wave in (7):

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{lm} i^l j_l(kx) Y_l^m(\hat{x}) Y_l^{m*}(\hat{k}) \overset{\substack{\vec{x}=-r\hat{n}\\ \hat{k}=\hat{z}}}{=} 4\pi \sum_{lm} i^l j_l(kr) \underbrace{Y_l^m(-\hat{n})}_{\substack{(-1)^l Y_l^m(\hat{n})\\ (Y4.10)}} \underbrace{Y_l^{m*}(\hat{z})}_{\substack{\sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}\\ (Y4.14)}}$$

$$\Rightarrow \underline{e^{-ikr\hat{n}\cdot\hat{z}} = \sqrt{4\pi} \sum_l (-i)^l \sqrt{2l+1}\, j_l(kr)\, Y_l^0(\hat{n})} \tag{8}$$

- The angular dependence of $\Theta(\eta_2,\hat{n})$ is a combination of the angular dependence of the plane wave, i.e. the Y_l^0 in (8), and the initial angular dependence at $\vec{x}_2 - r\hat{n}$, i.e. the Y_l^m in (7). The effect is mathematically like the addition of orbital (the plane wave) and spin (the initial) angular momentum into the total angular momentum.

- To get the multipoles $\Theta_l^m(\eta_2)$ of $\Theta(\eta_2,\hat{n})$, we need the total angular dependence of the line-of-sight mode functions $\qquad$ Use now CG series, Eq. (F6.11)

$$G_{l'k}^m(r,\hat{n}) = 4\pi \sum_l (-i)^{l'+l} \sqrt{\frac{2l+1}{2l'+1}}\, j_l(kr)\, \overbrace{Y_{l'}^m(\hat{n})\, Y_l^0(\hat{n})}$$

$$= 4\pi \sum_l (-i)^{l'+l} \sqrt{\frac{2l+1}{2l'+1}}\, j_l(kr) \sum_j \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2j+1)}} \langle l0l'm|jm\rangle \langle l0l'0|j0\rangle\, Y_j^m(\hat{n})$$

$$= \sum_{lj} (-i)^{l'+l} (2l+1) \sqrt{\frac{4\pi}{2j+1}} \langle l0l'm|jm\rangle \langle l0l'0|j0\rangle\, j_l(kr)\, Y_j^m(\hat{n}) \tag{9}$$

where l represents the plane-wave, l' the initial and j the total angular dependence.

- Thus the new brightness function, $\Theta(\eta_2,\hat{n})$ at Eq.(6) can be written

$$\Theta(\eta_2,\hat{n}) = \sum_{l'm} \Theta_{l'}^{m}(\eta_1)\, G_{l'k}^{m}(r,\hat{n})$$

$$= \sum_{jm} \underbrace{\sum_{l'L} \Theta_{l'}^{m}(\eta_1)(-i)^{l'+L}(2L+1)\sqrt{\frac{4\pi}{2j+1}}\langle L0l'm|jm\rangle\langle L0l'0|j0\rangle\, j_L(kr)}_{(-i)^j\sqrt{\frac{4\pi}{2j+1}}\,\Theta_j^m(\eta_2)} Y_j^m(\hat{n}) \qquad (10)$$

where all the angular dependence is in the $Y_j^m(\hat{n})$, and thus we can directly pick up the new multipoles

$$\underline{\Theta_j^m(\eta_2) = \sum_{l'L}(-i)^{l'+L-j}(2L+1)\langle L0l'm|jm\rangle\langle L0l'0|j0\rangle\, j_L(k\eta_2-k\eta_1)\,\Theta_{l'}^{m}(\eta_1)} \qquad (11)$$

This equation integrates the free streaming from η_1 to η_2.

- The sum over L produces a radial function which is a linear combination of spherical Bessel functions j_L. We define such new radial functions $j_j^{(l'm)}$ so that the line-of-sight mode functions $G_{l'k}^{m}$ can be written as

$$G_{l'k}^{m}(r,\hat{n}) \equiv \sqrt{4\pi}\sum_j (-i)^j\sqrt{2j+1}\; j_j^{(l'm)}(kr)\, Y_j^m(\hat{n}) \qquad (12)$$

in analogy of $(\vec{x}=-r\hat{n})$

$$e^{i\vec{k}\cdot\vec{x}} = \sqrt{4\pi}\sum_j(-i)^j\sqrt{2j+1}\; j_j(kr)\, Y_j^0(\hat{n}) . \qquad (13)$$

From (9) we have that

$$\boxed{j_j^{(l'm)}(kr) \equiv \sum_L (-i)^{L+l'-j}\left(\frac{2L+1}{2j+1}\right)\langle L0l'm|jm\rangle\langle L0l'0|j0\rangle\, j_L(kr)} \qquad (14)$$

and (11) becomes

$$\boxed{\Theta_j^m(\eta_2) = (2j+1)\sum_{l'} j_j^{(l'm)}(k\eta_2-k\eta_1)\,\Theta_{l'}^{m}(\eta_1)} \qquad (15)$$

where we have so far included only the free streaming effect.

- The radial functions $j_j^{(l'm)}$ turn out to be real (since $\langle L0l'0|j0\rangle = 0$ for $l'+L-j$ odd).

- We now have to add also the effect of the loss and source terms:

- The loss term is easy to integrate,

$$\Theta_l^{m\,\prime} = -a n_e \sigma_T \Theta_l^m \quad \Rightarrow \quad d \ln \Theta_l^m = -a n_e \sigma_T \, d\eta$$

$$\Rightarrow \ln \frac{\Theta_l^m(\eta_2)}{\Theta_l^m(\eta_1)} = -\int_{\eta_1}^{\eta_2} a n_e \sigma_T \, d\eta \quad \Rightarrow \quad \Theta_l^m(\eta_2) = e^{-\tau(\eta_1,\eta_2)} \, \Theta_l^m(\eta_1) \qquad (16)$$

where the background quantity

$$\boxed{\tau(\eta_1,\eta_2) \equiv \int_{\eta_1}^{\eta_2} a n_e \sigma_T \, d\eta} \qquad (17)$$

is the optical depth from η_1 to η_2.

Eq. (16) represents the damping of the anisotropy due to the Thomson scattering loss. Combining it with the free streaming effect of (15), we have

$$\Theta_j^m(\eta_2) = (2j+1)\, e^{-\tau(\eta_1,\eta_2)} \sum_{l'} \Theta_{l'}^m(\eta_1)\, j_j^{(l'm)}(k\eta_2 - k\eta_1) \qquad (18)$$

which represents how the anisotropy present at an earlier time η_1 is damped and transported by free streaming to a later time η_2.

- However, between η_1 and η_2, new anisotropy is also generated by the source term,

$$\frac{d\Theta_l^m(\eta)}{d\eta} = S_l^m(\eta) \qquad (19)$$

After being generated at time $\eta \in (\eta_1, \eta_2)$, this new anisotropy is also damped and transported by free streaming, but only from time η to η_2. Thus this adds a new term to Eq. (18),

$$\boxed{\begin{aligned} \Theta_j^m(\eta_2) = \; & (2j+1)\, e^{-\tau(\eta_1,\eta_2)} \sum_{l'} \Theta_{l'}^m(\eta_1)\, j_j^{(l'm)}(k\eta_2 - k\eta_1) \\ & + (2j+1) \int_{\eta_1}^{\eta_2} d\eta \; e^{-\tau(\eta,\eta_2)} \sum_{l'} S_{l'}^m(\eta)\, j_j^{(l'm)}(k\eta_2 - k\eta) \end{aligned}} \qquad (20)$$

This is the full result, that evolves the multipoles Θ_l^m from η_1 to η_2. Note that the sum for the source terms $S_{l'}^m$ goes only to $l'=2$, since there are no higher source terms.

- Towards early times the rate of scatterings $an_e\sigma_T$ increases, and as $\eta_1 \to 0$ (or in practice, to some time sufficiently long before photon decoupling),

$$\tau(\eta_1,\eta_2) \to \infty \qquad \Rightarrow \qquad e^{-\tau(\eta_1,\eta_2)} \to 0 \tag{21}$$

so that the original anisotropy at time η_1 is completely damped away, and all the anisotropy at η_2 comes from the source terms.

- Thus we can write for the present-day brightness function

$$\boxed{\Theta_j^m(\eta_0) = (2j+1)\int_0^{\eta_0} d\eta\, e^{-\tau(\eta)} \sum_{l'} S_{l'}^m(\eta)\, j_j^{(l'm)}(k\eta_0 - k\eta)} \tag{22}$$

where $\underline{\tau(\eta) \equiv \tau(\eta,\eta_0)}$ is the optical depth from time η to present time. (23)

- Eq. (22) is the line-of-sight integral for $\Theta_l^m(\eta_0)$. It is a formal solution of Eq. (1.1a) which can be verified by direct substitution. (Doing this is some work; it involves the properties of spherical Bessel functions $j_l(x)$, like recursion relations and values at $x=0$.) Note that the sources $S_l^m(\eta)$ include $\Theta_0^0(\eta)$, $\Theta_2^m(\eta)$, and $E_2^m(\eta)$. Thus Eq. (22) will not by itself solve Eq. (1.1a). We still need to integrate Eq. (1) numerically for the low l to get the sources $S_l^m(\eta)$. The point of Eq. (22) is that it gives all the higher l from these low multipoles.

- Now in (22) & (14) the l' represents the source, l the plane-wave, and j the total angular dependence; and the radial functions $j_l^{(l'm)}$ are needed only for $l' = 0,1,2$ and $m = -l',..,l'$, but they are needed for all l up to $l = j_{max}+2$, where j_{max} is the highest multipole we are interested in.

- For (22), we only need those radial functions $j_j^{(l'm)}$ for which $S_{l'}^m$ are nonzero.

L2.2 Polarization

- Polarization is handled the same way. First the free streaming, which gives

$$(Q \pm iU)(y_2, \vec{x}_2, \hat{n}) = (Q \pm iU)(y_1, \vec{x}_2 - r\hat{n}, \hat{n}) \tag{24}$$

The multipole expansion of (2) is

$$(Q \pm iU)(y, \vec{x}, \hat{n}) = \sum_{lm} (-i)^l \sqrt{\frac{4\pi}{2l+1}} \left[E_l^m(y) \pm iB_l^m(y)\right] {}_{\pm 2}Y_l^m(\hat{n}) e^{i\vec{k}\cdot\vec{x}}$$

$$\equiv \sum_{lm} \left[E_l^m(y) \pm iB_l^m(y)\right] {}_{\pm 2}G_{l\vec{k}}^m(\vec{x}, \hat{n}) \tag{25}$$

where we have defined the polarization mode functions ${}_{\pm 2}G_{l\vec{k}}^m$ as

$${}_{\pm 2}G_{l\vec{k}}^m(\vec{x}, \hat{n}) \equiv (-i)^l \sqrt{\frac{4\pi}{2l+1}} \, {}_{\pm 2}Y_l^m(\hat{n}) e^{i\vec{k}\cdot\vec{x}} \tag{26}$$

Written in terms of the mode functions, Eq. (24) becomes

$$(Q \pm iU)(y_2, \vec{x}, \hat{n}) = \sum_{lm} \left[E_l^m(y_1) \pm iB_l^m(y_1)\right] {}_{\pm 2}G_{l\vec{k}}^m(\vec{x}_2 - r\hat{n}, \hat{n}) \tag{27}$$

where $${}_{\pm 2}G_{l\vec{k}}^m(\vec{x}_2 - r\hat{n}, \hat{n}) = (-i)^l \sqrt{\frac{4\pi}{2l+1}} \, {}_{\pm 2}Y_l^m(\hat{n}) e^{i\vec{k}\cdot\vec{x}_2} e^{-ikr\hat{z}\cdot\hat{n}} \equiv e^{i\vec{k}\cdot\vec{x}_2} \, {}_{\pm 2}G_{lk}^m(r, \hat{n}) \tag{28}$$

and $${}_{\pm 2}G_{lk}^m(r, \hat{n}) \equiv {}_{\pm 2}G_{l\vec{k}}^m(-r\hat{n}, \hat{n}) = (-i)^l \sqrt{\frac{4\pi}{2l+1}} \, {}_{\pm 2}Y_l^m(\hat{n}) e^{-ikr\hat{z}\cdot\hat{n}} \tag{29}$$

is the line-of-sight polarization mode function.

- Now we just need to pick the multipoles $E_l^m(y_2)$, $B_l^m(y_2)$ of $(Q \pm iU)(y_2, \hat{n})$ from

$$(Q \pm iU)(y_2, \vec{x}_2, \hat{n}) = e^{i\vec{k}\cdot\vec{x}_2} \overbrace{\sum_{lm} \left[E_l^m(y_2) \pm iB_l^m(y_2)\right] \underbrace{(-i)^l \sqrt{\frac{4\pi}{2l+1}} \, {}_{\pm 2}Y_l^m(\hat{n}) e^{-ikr\hat{z}\cdot\hat{n}}}_{{}_{\pm 2}G_{lk}^m(r, \hat{n})}}^{(Q \pm iU)(y_2, \hat{n})} \tag{30}$$

Using the expansion (8) of the plane wave, this becomes

$$(Q \pm iU)(y_2, \hat{n}) = \sum_{l'm} \left[E_{l'}^m(y_1) \pm iB_{l'}^m(y_1)\right] \underbrace{(-i)^{l'} \sqrt{\frac{4\pi}{2l+1}} \cdot \sqrt{4\pi} \sum_l (-i)^l \sqrt{2l+1} \, j_l(kr) Y_l^0(\hat{n}) \cdot {}_{\pm 2}Y_{l'}^m(\hat{n})}_{{}_{\pm 2}G_{l'k}^m(r, \hat{n})} \tag{31}$$

- To pick out the multipoles of Eq. (31), we need to express the total angular dependence of ${}_{\pm2}G_{l'k}^{m}(r,\hat{n})$ as a linear combination of ${}_{s}Y_{j}^{m}(\hat{n})$. We use the CG series for spin-s harmonics (Eq. C5.5), which gives

$$Y_{l}^{0}(\hat{n})\,{}_{s}Y_{l'}^{m}(\hat{n}) = \sqrt{\frac{(2l+1)(2l'+1)}{4\pi}}\sum_{j}\frac{1}{\sqrt{2j+1}}\langle l0l'm|jm\rangle\langle l0;l',-s|j,-s\rangle\,{}_{s}Y_{j}^{m}(\hat{n}) \qquad (32)$$

to write

$${}_{s}G_{l'k}^{m}(r,\hat{n}) = \sum_{lj}(-i)^{l'+l}(2l+1)\sqrt{\frac{4\pi}{2j+1}}\langle l0l'm|jm\rangle\langle l0;l',-s|j,-s\rangle\, j_{l}(kr)\,{}_{s}Y_{j}^{m}(\hat{n}). \qquad (33)$$

With (33), Eq. (31) becomes

$$(Q\pm iU)(y_2,\hat{n}) = \sum_{jm}\underbrace{\sum_{ll'}\left[E_{l'}^{m}(y_1)\pm iB_{l'}^{m}(y_1)\right](-i)^{l+l'}(2l+1)\sqrt{\frac{4\pi}{2j+1}}\langle l0l'm|jm\rangle\langle l0;l',\mp2|j,\mp2\rangle j_{l}(kr)}_{(-i)^{j}\sqrt{\frac{4\pi}{2j+1}}\left[E_{j}^{m}(y_2)\pm iB_{j}^{m}(y_2)\right]}\,{}_{\pm2}Y_{j}^{m}(\hat{n}) \qquad (34)$$

from which we can directly pick up the polarization multipoles

$$\underline{E_{j}^{m}(y_2)\pm iB_{j}^{m}(y_2) = \sum_{ll'}(-i)^{l+l'-j}(2l+1)\langle l0l'm|jm\rangle\langle l0;l',\mp2|j,\mp2\rangle\, j_{l}(kr)\left[E_{l'}^{m}(y_1)\pm iB_{l'}^{m}(y_1)\right]} \qquad (35)$$

This equation integrates the free streaming of polarization from y_1 to y_2.

(To get the E_{j}^{m} and B_{j}^{m} just take the sum and the difference of the $\pm$ cases).

- Again, we define new radial functions ${}_{\pm2}j_{j}^{(l'm)}$ to correspond to the linear combinations of j_l (sum over l) that appear in (35) for each l' term:

$$\underline{{}_{\pm2}j_{j}^{(l'm)}(kr) \equiv \sum_{l}(-i)^{l+l'-j}\left(\frac{2l+1}{2j+1}\right)\langle l0l'm|jm\rangle\langle l0;l',\mp2|j,\mp2\rangle\, j_{l}(kr)} \qquad (36)$$

so that

$${}_{\pm2}G_{l'k}^{m}(r,\hat{n}) = \sqrt{4\pi}\sum_{j}(-i)^{j}\sqrt{2j+1}\;{}_{\pm2}j_{j}^{(l'm)}(kr)\;{}_{\pm2}Y_{l}^{m}(\hat{n}) \qquad (37)$$

and

$$\underline{E_{j}^{m}(y_2)\pm iB_{j}^{m}(y_2) = (2j+1)\sum_{l'}{}_{\pm2}j_{j}^{(l'm)}(ky_2-ky_1)\left[E_{l'}^{m}(y_1)\pm iB_{l'}^{m}(y_1)\right]} \qquad (38)$$

- Including thus the effect of the loss term, $\frac{d}{dy}(E_\ell^m \pm iB_\ell^m) = -an_e\sigma_T \cdot (E_\ell^m \pm iB_\ell^m)$ (39)

and the source term, $\frac{d}{dy}(E_\ell^m \pm iB_\ell^m) = Z_\ell^m$, we get the line-of-sight integral (40)

from y_1 to y_2 (in analogy to Eq. (20)) as

$$\boxed{\begin{aligned} E_j^m(y_2) \pm iB_j^m(y_2) &= (2j+1)\, e^{-\tau(y_1,y_2)} \sum_{\ell'} \left[E_{\ell'}^m(y_1) \pm iB_{\ell'}^m(y_1)\right] {}_{\pm2}j_j^{(\ell' m)}(ky_2-ky_1) \\ &+ (2j+1)\int_{y_1}^{y_2} dy\, e^{-\tau(y_1,y_2)}\, Z_2^m(y)\, {}_{\pm2}j_j^{(2m)}(ky_2-ky_1) \end{aligned}} \tag{41}$$

Note that there is no sum over ℓ' for the source terms $Z_{\ell'}^m$, since the only nonzero ones have $\ell' = 2$.

- For the present-day polarization multipoles we thus get

$$E_j^m(y_0) \pm iB_j^m(y_0) = (2j+1)\int_0^{y_0} dy\, e^{-\tau(y)}\, Z_2^m(y)\, {}_{\pm2}j_j^{(2m)}(ky_0-ky) \tag{42}$$

- It turns out (§L3) that the ${}_{\pm2}j_j^{(2m)}$ are complex conjugates of each other

$${}_{-2}j_j^{(2m)} = {}_{+2}j_j^{(2m)*} \tag{43}$$

so that they can be written as $\underline{{}_{\pm2}j_j^{(2m)} \equiv \varepsilon_j^m \pm i\beta_j^m}$ (44)

where the radial functions ε_j^m and β_j^m are real. Thus we obtain from (42)

$$\boxed{\begin{aligned} E_\ell^m(y_0) &= (2j+1)\int_0^{y_0} dy\, e^{-\tau(y)} \left[-\sqrt{6}\, an_e\sigma_T P^{(m)}(y)\right] \varepsilon_\ell^m(ky_0-ky) \quad (45a) \\ B_\ell^m(y_0) &= (2j+1)\int_0^{y_0} dy\, e^{-\tau(y)} \left[-\sqrt{6}\, an_e\sigma_T P^{(m)}(y)\right] \beta_\ell^m(ky_0-ky) \quad (45b) \end{aligned}}$$

Note that the only difference between (45a) and (45b) is the radial functions. E and B modes have the same source, and the relation between E and B mode polarization is determined completely by the structure of the ε_ℓ^m and β_ℓ^m radial functions (and its relation to the time dependence of $P^{(m)}(y)$). They are needed for $m = 0, \pm1, \pm2$, corresponding to scalar, vector, and tensor perturbations.