L. LINE-OF-SIGHT INTEGRATION

L1. Boltzmann Equations

- In Chapter §C we obtained the full photon Boltzmann equations in multipole space. They can be written as

$$\boxed{\begin{aligned}
\Theta_\ell^{m\,\prime} &= \frac{\sqrt{\ell^2-m^2}}{2\ell-1}\,k\Theta_{\ell-1}^m - \frac{\sqrt{(\ell+1)^2-m^2}}{2\ell+3}\,k\Theta_{\ell+1}^m - an_e\sigma_T\Theta_\ell^m + S_\ell^m \\
E_\ell^{m\,\prime} &= \frac{\sqrt{(\ell^2-m^2)(\ell^2-4)}}{\ell(2\ell-1)}\,kE_{\ell-1}^m - \frac{2m}{\ell(\ell+1)}\,kB_\ell^m - \frac{\sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+1)^2-4}}{\ell(2\ell+3)}\,kE_{\ell+1}^m - an_e\sigma_T E_\ell^m + Z_\ell^m \\
B_\ell^{m\,\prime} &= \frac{\sqrt{(\ell^2-m^2)(\ell^2-4)}}{\ell(2\ell-1)}\,kB_{\ell-1}^m + \frac{2m}{\ell(\ell+1)}\,kE_\ell^m - \frac{\sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+1)^2-4}}{\ell(2\ell+3)}\,kB_{\ell+1}^m - an_e\sigma_T B_\ell^m
\end{aligned}} \qquad (1)$$

where we have grouped everything else except the free streaming and loss terms into the S_ℓ^m and Z_ℓ^m, which we shall call source terms. The only nonzero source terms are

$$\boxed{\begin{aligned}
S_0^0 &= an_e\sigma_T\Theta_0^0 + \psi' & & \\
S_1^0 &= an_e\sigma_T V_b^{(0)} + k\phi & S_1^{\pm1} &= an_e\sigma_T\cdot V_b^{(\pm1)} + B^{(\pm1)\prime} \\
S_2^0 &= an_e\sigma_T P^{(0)} & S_2^{\pm1} &= an_e\sigma_T\cdot P^{(\pm1)} \\
S_2^{\pm2} &= an_e\sigma_T P^{(\pm2)} - h^{(\pm2)\prime} & Z_2^m &= -\sqrt{6}\,an_e\sigma_T\cdot P^{(m)}
\end{aligned}} \qquad (2)$$

where $\underline{P^{(m)} \equiv \frac{1}{10}\left(\Theta_2^m - \sqrt{6}\,E_2^m\right)}$ (3)

and $\boxed{\begin{aligned} V_b^{(0)} &\equiv iV_b^3 \\ V_b^{(\pm1)} &\equiv \mp\frac{i}{\sqrt{2}}\left(V_b^1 \mp iV_b^2\right)\end{aligned}}$ $\qquad \underline{B^{(\pm1)} \equiv \mp\frac{i}{\sqrt{2}}\left(B_1 \mp iB_2\right)}$ (4)

N.B. we changed the phase convention for $V_b^{(m)}$ and $B^{(m)}$ by i from Eqs (F7.8) and (C7.7) to agree w Hu&White. The $'$ represent $\frac{d}{d\eta}$.

- Note that Eq. (1) contains equations for arbitrarily high ℓ, but only for $m = 0, \pm1, \pm2$.

- The source terms contain the gravitational effects ψ', $k\phi$, $B^{(\pm1)'}$, $-h^{(\pm2)'}$, the Doppler effect $a n_e \sigma_T \cdot V_b^{(m)}$, and the Thomson scattering gain terms $a n_e \sigma_T \cdot \Theta_0^0$ and $a n_e \sigma_T \cdot P^{(m)}$. $P^{(m)}$ represents the effect of the anisotropy of Compton scattering.

  The source terms generate anisotropy and polarization at low multipoles $l = 0, 1, 2$. Free streaming then transports these effects to higher multipoles, and the Thomson scattering loss terms damp the anisotropy and polarization.

- Eq. (1) is an infinite hierarchy of differential equations, the Boltzmann hierarchy. In practice, the Boltzmann hierarchy can be truncated: We decide how high l we are interested in, and then choose some higher $l = l_{max}$ up to which we calculate the Θ_l^m, E_l^m, B_l^m. For $l > l_{max}$ they are just assumed zero and not included in the equations. Eq. (1) up to $l = l_{max}$ can then be numerically integrated together with the Einstein equations for metric perturbations and the energy-momentum continuity equations for baryons and cold dark matter. For neutrinos there is a similar Boltzmann hierarchy as for photons, except that there are no collision terms.

  Because of the truncation, the integration will give wrong results for the highest multipoles near l_{max}, but using some sophistication in the truncation, to prevent a reflection effect at l_{max}, the effect of the truncation can be confined to a small number of l near l_{max}.

  Note that the integration has to be done separately for each scale k; in practice one chooses a number of k values that sample the scales of interest sufficiently densely.

- Thus one can obtain the present-day angular power spectra C_ℓ^{TT}, C_ℓ^{EE}, C_ℓ^{BB}, C_ℓ^{TE}, by starting with some initial conditions at some early time well before recombination, and integrating Eqs. (1) numerically all the way to the present time. This is how the first "Boltzmann codes" for accurate calculation of C_ℓ worked. The problem with this approach is, that the computation is quite time-consuming, since we are interested in C_ℓ up to ℓ of several thousand. Thus one has to integrate a system of several thousand differential equations.

- Zaldarriaga and Seljak (1996, 1997) came up with a better approach, called Line-of-Sight Integration. It is based on the observation that the free streaming effect, which looks complicated in the Fourier & multipole space (Eq. 1) is very simple in the $\vec{x}, \hat{n}$ space; and can be integrated directly, together with the loss terms. Thus the Boltzmann hierarchy is needed only for the lowest multipoles, to get correctly the evolution of the source terms S_ℓ^m and Z_2^m.