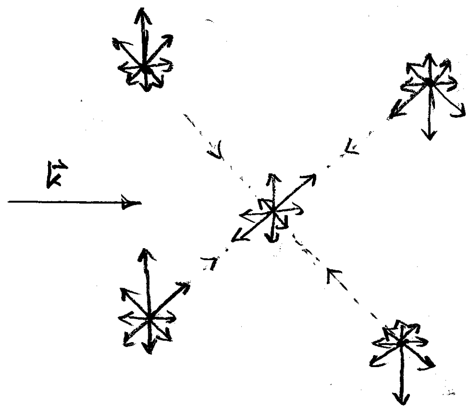
F8. Free Streaming

- We now take up the "free-streaming" part of the brightness eq.,

$$\frac{\partial \Theta}{\partial \eta} = -ik n_3 \Theta \qquad \text{where} \quad \Theta = \Theta(\eta, \hat{n}) \qquad (1)$$

- The physics of this is the mixing of the photon distribution at different locations, as the photons move from one place to another.



The figure shows the generation of quadrupole anisotropy ($L=2$) – at the center of the fig, which represents a later time η – from dipole anisotropy ($L=1$).

- Eq. (1) differs from the previous case (metric perturbations, F7) in that now $\Theta(\hat{n})$, with its own direction dependence appears also on the rhs; besides the direction dependence from n_3. This leads to products of Y_L^m for which we need the CG series, (Eq. 6.12).

$$\frac{\partial \Theta}{\partial \eta} = -ik\cos\vartheta \cdot \Theta(\hat{n}) = -ik \cdot \sqrt{\frac{4\pi}{3}}\, Y_1^0(\hat{n}) \cdot \Theta(\hat{n}) = -ik\sqrt{\frac{4\pi}{3}} \sum_L \Theta_L^m (-i)^L \sqrt{\frac{4\pi}{2L+1}}\, Y_1^0 Y_L^m$$

$$= -ik \sum_L \Theta_L^m (-i)^L \sqrt{\frac{4\pi}{2L+1}} \cdot \left[\sqrt{\frac{(L+m)(L-m)}{(2L-1)(2L+1)}}\, Y_{L-1}^m + \sqrt{\frac{(L+1+m)(L+1-m)}{(2L+1)(2L+3)}}\, Y_{L+1}^m \right]$$

Since we sum over all L, we can take the sums over the two terms separately, write in the first one $L = L'+1$ and in the second one $L = L'-1$, and consider them as sums over L',

$$= -ik \sum_{L'} \Theta_{L'+1}^m (-i)^{L'+1} \sqrt{\frac{4\pi}{2L'+3}} \sqrt{\frac{(L'+1+m)(L'+1-m)}{(2L'+1)(2L'+3)}}\, Y_{L'}^m \; - ik \sum_{L'} \Theta_{L'-1}^m (-i)^{L'-1} \sqrt{\frac{4\pi}{2L'-1}} \sqrt{\frac{(L'+m)(L'-m)}{(2L'-1)(2L'+1)}}\, Y_{L'}^m .$$

and then drop the primes. We can now pick the multipoles

$$\underline{\Theta_L^{m\,\prime}} = i^L \sqrt{\frac{2L+1}{4\pi}} \int d\Omega\, Y_L^{m*} \cdot \frac{\partial \Theta}{\partial \eta}$$

$$= -ik \cdot i^L \sqrt{\frac{\cancel{2L+1}}{\cancel{4\pi}}} \left[\Theta_{L+1}^m (-i)^{L+1} \sqrt{\frac{\cancel{4\pi}}{2L+3}} \sqrt{\frac{(L+1+m)(L+1-m)}{(\cancel{2L+1})(2L+3)}} + \Theta_{L-1}^m (-i)^{L-1} \sqrt{\frac{\cancel{4\pi}}{2L-1}} \sqrt{\frac{(L+m)(L-m)}{(2L-1)(\cancel{2L+1})}} \right]$$

$$= \underline{-k \frac{\sqrt{(L+1)^2 - m^2}}{2L+3}\, \Theta_{L+1}^m + k \frac{\sqrt{L^2 - m^2}}{2L-1}\, \Theta_{L-1}^m} \qquad (2)$$

- You might worry about what happens at $l=0$; but the formulae work nicely so that for each "nonexistent" Y_l^m, i.e. $|m|>l$ or $l<0$, that seem to appear in the eqs, the coefficient is always $=0$. For example, Eq. (6.12) for $l=0$ becomes just

$$Y_1^0 \cdot Y_0^0 = 0 + \sqrt{\frac{3}{4\pi}}\sqrt{\frac{1\cdot 1}{1\cdot 3}}\, Y_1^0 = \frac{1}{\sqrt{4\pi}} Y_1^0$$

and Eq. (2) for $l=0$ becomes

$$\Theta_0^{0\prime} = -k\cdot\frac{1}{3}\Theta_1^0 + 0 = -\frac{k}{3}\Theta_1^0 . \qquad (3)$$

- From Eq. (2) we see that the effect of free streaming is to mix the l modes, without mixing the m modes.

- Combining the results from §F7 (metric perturbations) and §F8 (free streaming), we have that the multipole expansion of the collisionless brightness equation (5.4) is

$$\begin{aligned}
\Theta_0^{0\prime} &= -\frac{k}{3}\Theta_1^0 + \psi' \\
\Theta_1^{0\prime} &= -\frac{2}{5}k\,\Theta_2^0 + k\Theta_0^0 + k\phi \\
\Theta_1^{\pm1\prime} &= -\frac{\sqrt{3}}{5}k\,\Theta_2^{\pm1} + iB^{(\pm1)\prime} \\
\Theta_2^{0\prime} &= -\frac{3}{7}k\Theta_3^0 + \frac{2}{3}k\Theta_1^0 \\
\Theta_2^{\pm1\prime} &= -\frac{\sqrt{8}}{7}k\Theta_3^{\pm1} + \frac{1}{\sqrt{3}}k\Theta_1^{\pm1} \\
\Theta_2^{\pm2\prime} &= -\frac{\sqrt{5}}{7}k\Theta_3^{\pm2} - h^{(\pm2)\prime} \\
\Theta_l^{m\prime} &= -\frac{\sqrt{(l+1)^2-m^2}}{2l+3}\cdot k\Theta_{l+1}^m + \frac{\sqrt{l^2-m^2}}{2l-1}k\Theta_{l-1}^m \quad \text{for } l\geq 3
\end{aligned} \qquad (4)$$

Metric perturbations generate CMB anisotropy at low l, from where the free streaming transports it up the "l ladder". Since m modes are not mixed; we still have just $m=0$ modes for the case of scalar perturbations, $m=\pm1$ for vector, and $m=\pm2$ for tensor perturbations.

Thus we will never get any $|m|\geq 3$ modes!

(Of course the whole discussion is in context of 1st order perturbation theory.)