F7. Generation of CMB Anisotropy by Metric Perturbations

- Consider now the redshift part of the brightness equation (5.4):

$$\frac{\partial\Theta}{\partial y} = -\tfrac{1}{2}h_{ij}'n^in^j - h_{0i}'n^i + \tfrac{1}{2}ikn_3h_{00} \qquad (1) \qquad\qquad \Theta = \Theta(y,\hat{n})$$

This tells us how metric perturbations affect the brightness function.

We consider scalar, vector, and tensor perturbations separately.

F7.1 Scalar Perturbations

- We choose the Newtonian gauge $\Rightarrow$ $h_{ij} = -2\psi\delta_{ij}$, $h_{0i} = 0$, $h_{00} = -2\phi$

Eq. (1) becomes

$$\frac{\partial\Theta}{\partial y} = +\psi' - ikn_3\phi = \psi'\cdot\sqrt{4\pi}\,Y_0^0(\hat{n}) - ik\phi\cdot\sqrt{\tfrac{4\pi}{3}}\,Y_1^0(\hat{n})$$

For the multipoles we get

$$a_{lm}' = \int d\Omega\, Y_l^{m*}(\hat{n})\frac{\partial\Theta}{\partial y} = \psi'\cdot\sqrt{4\pi}\cdot\delta_l^0\delta_m^0 - ik\phi\cdot\sqrt{\tfrac{4\pi}{3}}\cdot\delta_l^1\delta_m^0$$

$$\Rightarrow \quad \Theta_l^{m\,\prime} = i^l\sqrt{\frac{2l+1}{4\pi}}\,a_{lm}' = \psi'\cdot\delta_l^0\delta_m^0 + k\phi\cdot\delta_l^1\delta_m^0 \qquad (2)$$

So for the different multipoles we have

$$\boxed{\begin{aligned}\Theta_0^{0\,\prime} &= \psi' \\ \Theta_1^{0\,\prime} &= k\phi\end{aligned}} \qquad (3)$$

No effect on the other Θ_l^m.

Eq. (3) corresponds to $\delta_\gamma' = 4\psi'$, $v_\gamma' = k\phi$ in the energy tensor description (exercise).

(3a) represents the change in photon energy density due to the extra expansion of space ψ'.

(3b) represents the effect on photons by the gravitational potential gradient $k\phi$.

F7.2 Vector Perturbations

- For vector perturbations $$h_{\mu\nu} = \begin{bmatrix} 0 & -B_i \\ -B_i & 2E_{ij}^V \end{bmatrix} = \begin{bmatrix} 0 & -B_1 & -B_2 & 0 \\ -B_1 & & & -iE_1 \\ -B_2 & & & -iE_2 \\ 0 & -iE_1 & -iE_2 & 0 \end{bmatrix}$$

The gauge transformation is

$$\tilde{B}_i = B_i + \xi^{i\prime} \quad \text{and} \quad \tilde{E}_i = E_i + k\xi^i$$

We choose the gauge where $E_i = 0$ (by taking $\xi^i = -\frac{1}{k}E_i$)

$$\Rightarrow \quad h_{00} = h_{ij} = 0, \quad h_{0i} = -B_i \tag{4}$$

- Eq. (1) becomes

$$\frac{\partial\Theta}{\partial y} = B_i' n_i = n_1 B_1' + n_2 B_2' = \sqrt{\frac{2\pi}{3}}\left[(-Y_1^1 + Y_1^{-1})B_1' + i(Y_1^1 + Y_1^{-1})B_2'\right]$$

$$= \sqrt{\frac{4\pi}{3}}\left[-\frac{1}{\sqrt{2}}(B_1' - iB_2')Y_1^1 + \frac{1}{\sqrt{2}}(B_1' + iB_2')Y_1^{-1}\right] = \sqrt{\frac{4\pi}{3}}\left[B^{(1)\prime}Y_1^1 + B^{(-1)\prime}Y_1^{-1}\right]$$

where we defined

$$B^{(1)} \equiv \frac{-1}{\sqrt{2}}(B_1 - iB_2), \qquad B^{(-1)} \equiv \frac{1}{\sqrt{2}}(B_1 + iB_2) \tag{6}$$

For the multipoles we get (only $L=1$ are nonzero)

$$\Theta_L^{m\prime} = i^L\sqrt{\frac{2L+1}{4\pi}}\int d\Omega\, Y_L^{m*}(\hat{n})\frac{\partial\Theta(\hat{n})}{\partial y} = i\sqrt{\frac{3}{4\pi}}\cdot\sqrt{\frac{4\pi}{3}}\left[B^{(1)\prime}\delta_L^1\delta_m^1 + B^{(-1)\prime}\delta_L^1\delta_m^{-1}\right] \tag{7}$$

So for the different multipoles we have

$$\boxed{\begin{aligned}\Theta_1^{1\prime} &= iB^{(1)\prime} \\ \Theta_1^{-1\prime} &= iB^{(-1)\prime}\end{aligned}} \tag{8}$$

No effect on the other Θ_L^m.

F7.3 Tensor Perturbations

- For tensor perturbations, $h_{00} = h_{0i} = 0$ and $h_{ij} = \begin{bmatrix} h_+ & h_\times & \\ h_\times & -h_+ & \\ & & 0 \end{bmatrix}$
- Eq. (1) becomes

$$\frac{\partial}{\partial y}\Theta(y,\hat{n}) = -\tfrac{1}{2}h'_{ij}n_i n_j = -\tfrac{1}{2}\left[h'_+(n_1^2-n_2^2) + h'_\times \cdot 2n_1 n_2\right]$$

$$= -\sqrt{\frac{2\pi}{15}}\left[(Y_2^2+Y_2^{-2})h'_+ + i(-Y_2^2+Y_2^{-2})h'_\times\right] = \sqrt{\frac{4\pi}{5}}\cdot\frac{1}{\sqrt{6}}\left[(-h'_+ + ih'_\times)Y_2^2 + (-h'_+ - ih'_\times)Y_2^{-2}\right]$$

$$= \sqrt{\frac{4\pi}{5}}\cdot\left[h^{(2)'}Y_2^2 + h^{(-2)'}Y_2^{-2}\right] \quad \text{where} \quad \begin{aligned} h^{(2)} &\equiv \frac{-1}{\sqrt{6}}(h_+ - ih_\times) \\ h^{(-2)} &\equiv \frac{-1}{\sqrt{6}}(h_+ + ih_\times) \end{aligned} \qquad (9)$$

For the multipoles we get (only $l=2$ nonzero)

$$\Theta_l^{m'} = i^l\sqrt{\frac{2l+1}{4\pi}}\int d\Omega\, Y_l^{m*}(\hat{n})\frac{\partial\Theta(\hat{n})}{\partial y} = -\sqrt{\frac{5}{4\pi}}\cdot\sqrt{\frac{4\pi}{5}}\left[h^{(2)'}\delta_l^2\delta_m^2 + h^{(-2)'}\delta_l^2\delta_m^{-2}\right] \qquad (10)$$

$$\Rightarrow \boxed{\begin{aligned} \Theta_2^{2'} &= -h^{(2)'} \\ \Theta_2^{-2'} &= -h^{(-2)'} \end{aligned}} \qquad (11)$$

No effect on the other Θ_l^m.

F7.4 Summary

We got that the gravitational redshift effect on the brightness function is

$$\begin{array}{|lll|} \Theta_0^{0'} = \psi' & \Theta_1^{1'} = iB^{(1)'} & \Theta_2^{2'} = -h^{(2)'} \\ \Theta_1^{0'} = k\phi & \Theta_1^{-1'} = iB^{(-1)'} & \Theta_2^{-2'} = -h^{(-2)'} \end{array} \qquad (12)$$

The important result is that:

scalar	perturbations	excite	$m=0$	modes
vector	perturbations	excite	$m=\pm1$	modes
tensor	perturbations	excite	$m=\pm2$	modes