F6. Spherical Harmonics

Explicit form $$Y_l^m(\vartheta,\varphi) \equiv (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \underbrace{P_l^m(\cos\vartheta)}_{\text{associated Legendre function, real}} e^{im\varphi} \tag{1}$$

Orthonormality $$\int d\Omega\, Y_l^m(\hat{n}) Y_{l'}^{m'}(\hat{n})^* = \delta_{ll'}\delta_{mm'} \tag{2}$$

Closure relation (addition theorem)$^{*)}$ $$\sum_m |Y_l^m(\vartheta,\varphi)|^2 = \frac{2l+1}{4\pi} \tag{3}$$

Legendre polynomials $$P_l(\cos\vartheta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^0(\vartheta,\varphi) \qquad \text{(independent of } \varphi\text{)} \tag{4}$$

Expansion of plane wave $$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{lm} i^l j_l(kx)\, Y_l^m(\hat{x})\, Y_l^m(\hat{k})^* \tag{5}$$

Explicit forms of $Y_l^m(\vartheta,\varphi) = Y_l^m(\hat{n})$ for $l \le 2$:

Here $\hat{n} = (n_1, n_2, n_3) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ is the direction unit vector

$$\begin{aligned}
Y_0^0 &= \frac{1}{\sqrt{4\pi}} \\
Y_1^1 &= -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi} &&= -\sqrt{\frac{3}{4\pi}} \cdot \frac{1}{\sqrt{2}}(n_1 + in_2) \\
Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos\vartheta &&= \sqrt{\frac{3}{4\pi}} \cdot n_3 \\
Y_1^{-1} &= \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-i\varphi} &&= \sqrt{\frac{3}{4\pi}} \cdot \frac{1}{\sqrt{2}}(n_1 - in_2) \\
Y_2^2 &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta\, e^{i2\varphi} &&= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot (n_1 + in_2)^2 \\
Y_2^1 &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cos\vartheta\sin\vartheta\, e^{i\varphi} &&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot n_3(n_1 + in_2) \\
Y_2^0 &= \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\vartheta - 1) &&= \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3n_3^2 - 1) \\
Y_2^{-1} &= \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cos\vartheta\sin\vartheta\, e^{-i\varphi} &&= \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot n_3(n_1 - in_2) \\
Y_2^{-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta\, e^{-i2\varphi} &&= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot (n_1 - in_2)^2
\end{aligned} \tag{6}$$

$$\Rightarrow \quad \begin{aligned}
n_1 &= \sqrt{\frac{2\pi}{3}}\,(-Y_1^1 + Y_1^{-1}) & n_1 n_2 &= i\sqrt{\frac{2\pi}{15}}\,(-Y_2^2 + Y_2^{-2}) \\
n_2 &= i\sqrt{\frac{2\pi}{3}}\,(Y_1^1 + Y_1^{-1}) & \tfrac{1}{2}(n_1^2 - n_2^2) &= \sqrt{\frac{2\pi}{15}}\,(Y_2^2 + Y_2^{-2}) \\
n_3 &= \sqrt{\frac{4\pi}{3}}\, Y_1^0
\end{aligned} \tag{7}$$

*) The addition theorem is $$\sum_m Y_l^m(\hat{n}')^* Y_l^m(\hat{n}) = \frac{2l+1}{4\pi} P_l(\cos\beta) = \frac{2l+1}{4\pi} P_l(\hat{n}'\cdot\hat{n})$$

where β is the angle between $\hat{n}'$ and $\hat{n}$.

Rotation of coordinate system

$$Y_l^{m'}(\vartheta',\varphi') = \sum_m Y_l^m(\vartheta,\varphi)\, D^l_{mm'}(\alpha,\beta,\gamma) \qquad (8)$$

What does this mean: On the sphere S^2 I can use many different spherical crd. systems (ϑ,φ). For example, the ecliptic and galactic crd. systems of the sky. Suppose that I have a function $f: S^2 \to \mathbb{C}$, $P \mapsto f(P)$, whose crd. representation in the (ϑ',φ') crd. system happens to be

(for $\forall P \in S^2$)

$f(P) = Y_l^{m'}(\vartheta',\varphi')$ for some particular l, m'. Then in another crd. system (ϑ,φ)

$$f(P) = \sum_m Y_l^m(\vartheta,\varphi) D^l_{mm'}(\alpha,\beta,\gamma) = Y_l^{m'}(\vartheta',\varphi')$$

Here ϑ,φ and ϑ',φ' are the crd's of the same point P in two different crd. systems.

α,β,γ are the Euler angles of the rotation of the crd. system $\vartheta,\varphi \to \vartheta',\varphi'$

$$D^l_{mm'}(\alpha,\beta,\gamma) = e^{-im\alpha}\, \underbrace{d^l_{mm'}(\beta)}_{\text{real}}\, e^{-im'\gamma}$$ are the Wigner D-functions (discussed more later).

The matrices $D^l(\alpha,\beta,\gamma)$ (with fixed l) form a representation of the rotation group.

Euler angles will be defined later when we actually do such rotations.

We can take (8) as the definition of the Wigner D-functions. The important point is that rotations mix different m, but do not mix different l.

A function $h: S^2 \to \mathbb{C}$ on the sphere can be expanded

$$h(\vartheta',\varphi') \equiv \sum_{lm'} a'_{lm'} Y_l^{m'}(\vartheta',\varphi')$$

$$h(\vartheta,\varphi) = \sum_{lmm'} a'_{lm'} Y_l^m(\vartheta,\varphi) D^l_{mm'}(\alpha,\beta,\gamma) \equiv \sum_{lm} a_{lm} Y_l^m(\vartheta,\varphi)$$

$$\Rightarrow \quad a_{lm} = \sum_{m'} D^l_{mm'}(\alpha,\beta,\gamma)\, a'_{lm'} \qquad (9)$$

(The Y_l^m form a basis for the set of functions on the sphere; the a_{lm} are the "components" of the function in this basis. Eq. (8) gives the transformation of the basis, and Eq. (9) the transformation of the components, in a rotation of the crd. system.)

- For later reference, the Y_l^m can be expressed in terms of (are special cases of) the $D^l_{mm'}$:

$$Y_l^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}}\, D^l_{-m,0}(\varphi,\vartheta,\chi) = \sqrt{\frac{2l+1}{4\pi}}\, D^l_{m0}(\varphi,\vartheta,\chi)^* \qquad (10)$$

(independent of χ)

<u>Product of two spherical harmonics</u> (the Clebsch-Gordan series)

$$Y_{l_1}^{m_1}(\vartheta,\varphi)\, Y_{l_2}^{m_2}(\vartheta,\varphi) = \sum_{jm} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi\,(2j+1)}}\, \langle l_1 0\, l_2 0 | j 0\rangle \langle l_1 m_1\, l_2 m_2 | jm\rangle\, Y_j^m(\vartheta,\varphi) \qquad (11)$$

where the Clebsch-Gordan coefficients*) $\langle l_1 m_1\, l_2 m_2 | jm\rangle = 0$ unless $m = m_1 + m_2$
and $j = |l_1 - l_2|, .., l_1 + l_2$

- A special case that we'll need soon is

$$Y_1^0(\vartheta,\varphi)\, Y_l^m(\vartheta,\varphi) = \sqrt{\frac{3}{4\pi}} \sqrt{\frac{(l+m)(l-m)}{(2l-1)(2l+1)}}\, Y_{l-1}^m + \sqrt{\frac{3}{4\pi}} \cdot \sqrt{\frac{(l+1+m)(l+1-m)}{(2l+1)(2l+3)}}\, Y_{l+1}^m \qquad (12)$$

which can also be derived from the recurrence relation for associated Legendre functions

$$\cos\vartheta \cdot P_l^m(\cos\vartheta) = \frac{l+m}{2l+1} P_{l-1}^m(\cos\vartheta) + \frac{l+1-m}{2l+1} P_{l+1}^m(\cos\vartheta)$$

*) Usually they are written as $\langle l_1 l_2\, m_1 m_2 | jm\rangle$. This notation reflects the VMK notation $C_{l_1 m_1 l_2 m_2}^{jm}$