F5. Fourier and Spherical Harmonic Expansion

- The brightness function $\Theta(\eta,\vec{x},\hat{n})$ has both position ($\vec{x}$) and direction ($\hat{n}$) dependence. We Fourier expand the position dependence,

$$\Theta(\eta,\vec{x},\hat{n}) = \sum_{\vec{k}} \Theta(\eta,\vec{k},\hat{n})\, e^{i\vec{k}\cdot\vec{x}} \tag{1}$$

The brightness equation (4.8) becomes

$$\frac{\partial\Theta}{\partial\eta} = -i\hat{n}\cdot\vec{k}\,\Theta - \tfrac{1}{2}h'_{ij}n^i n^j - h'_{0i}n^i + \tfrac{1}{2}i\hat{n}\cdot\vec{k}\,h_{00} \tag{2}$$

for $\Theta(\eta,\vec{k},\hat{n})$.

- We then expand the direction ($\hat{n}$) dependence of $\Theta(\eta,\vec{k},\hat{n})$ in spherical harmonics

$$\Theta(\eta,\vec{k},\hat{n}) = \sum_{lm} a_{lm}(\eta,\vec{k})\, Y_l^m(\hat{n}) \tag{3}$$

For the spherical harmonic expansion it turns out to be convenient to choose the z-axis parallel to the Fourier mode wave vector $\vec{k}$: $\hat{z} \parallel \vec{k}$.

Thus we use a different orientation of the background crd's for different $\vec{k}$ modes.

- It is simplest to think that we consider one Fourier mode $\vec{k}$; and then choose the orientation of the crd's, so that

$$\Theta(\eta,\vec{x},\hat{n}) = \Theta(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}} = \Theta(\eta,\hat{n})\, e^{ikz}$$

and

$$\frac{\partial\Theta}{\partial\eta} = -ikn_3\Theta - \tfrac{1}{2}h'_{ij}n^i n^j - h'_{0i}n^i + \tfrac{1}{2}ikn_3 h_{00} \tag{4}$$

For later convenience, we include a factor $i^l\sqrt{\frac{2l+1}{4\pi}}$ in the sph. harmonic (multipole) coefficients, and define

$$\Theta_l^m(\eta) \equiv i^l\sqrt{\frac{2l+1}{4\pi}}\, a_{lm} \equiv i^l\sqrt{\frac{2l+1}{4\pi}}\int d\Omega\, Y_l^{m*}(\hat{n})\,\Theta(\eta,\hat{n}) \tag{5}$$

- We'll now do the spherical harmonic expansion of (4) separately for redshift (F7) and free streaming (F8).

Properties of the spherical harmonics $Y_l^m(\hat{n}) = Y_l^m(\vartheta,\varphi)$ are discussed in F6.

- Note that $\Theta(\eta,\vec{x},\hat{n})$ is real $\Rightarrow$ $\Theta(\eta,-\vec{k},\hat{n}) = \Theta(\eta,\vec{k},\hat{n})^*$

Since $\Theta(\eta,\vec{k},\hat{n})$ is complex, there is no additional restriction on its multipoles Θ_m^l.

- We could have defined "unnormalized spherical harmonics"

$$\tilde{Y}_l^m(\hat{n}) \equiv (-i)^l \sqrt{\frac{4\pi}{2l+1}}\, Y_l^m(\hat{n}) = (-i)^l(-1)^m \sqrt{\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\vartheta)\, e^{im\varphi}$$

Many formulae are simpler for $\tilde{Y}_l^m(\hat{n})$ than for $Y_l^m(\hat{n})$, but the orthogonality relation is

$$\int d\Omega\, \tilde{Y}_l^m(\hat{n})^* \tilde{Y}_{l'}^{m'}(\hat{n}) = \frac{4\pi}{2l+1}\, \delta_l^{l'} \delta_m^{m'} .$$

The multipole expansion of the brightness function is now

$$\Theta(\eta,\hat{n}) = \sum \Theta_l^m(\eta)\, \tilde{Y}_l^m(\hat{n})$$

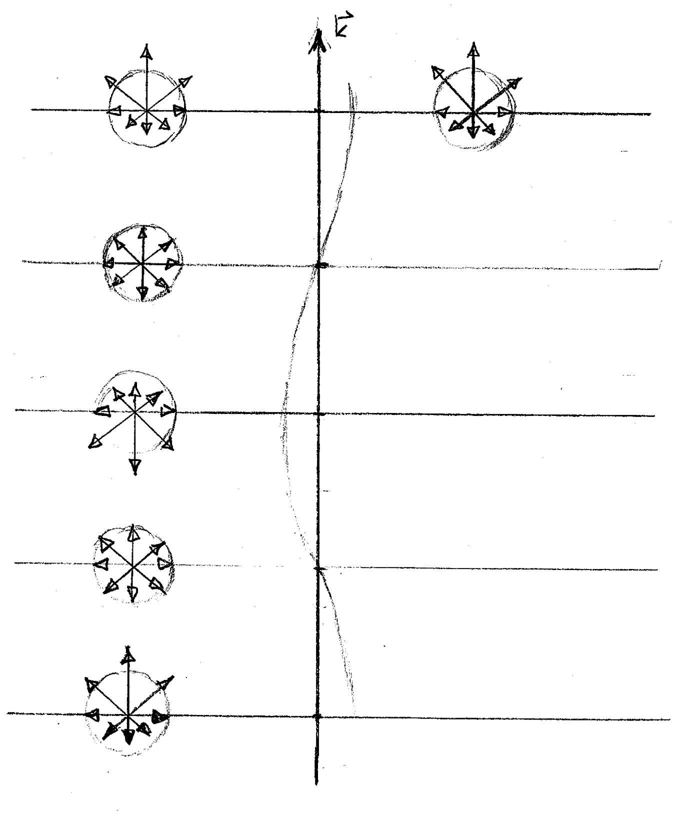
$$\Theta_l^m(\eta) = \frac{2l+1}{4\pi} \int d\Omega\, \tilde{Y}_l^{m*}(\hat{n})\, \Theta(\eta,\hat{n})$$

For the $m=0$ modes we have

$$\tilde{Y}_l^0(\hat{n}) = (-i)^l P_l(\cos\vartheta)$$

$$\Theta_l^0(\eta) = \frac{2l+1}{4\pi}\, i^l \int d\Omega\, P_l(\cos\vartheta)\,\Theta(\eta,\hat{n}) = (2l+1)\, i^l \int_{-1}^{1} \frac{d\cos\vartheta}{2}\, P_l(\cos\vartheta)\,\Theta(\eta,\hat{n})$$

- Figure of a Fourier mode w dipole ($l=1$) anisotropy:



The anisotropy $\Theta(\hat{n})$ is the same at every $\vec{x}$ except that it is modulated by the plane wave $e^{i\vec{k}\cdot\vec{x}}$ (in the figure as $\cos\vec{k}\cdot\vec{x}$, representing superposition of $\vec{k}$ and $-\vec{k}$ modes).