F4. The Brightness Equation

- Consider now the distribution function for photons. Since photons are massless,

$$E(\vec{q}) = q$$

- In thermal equilibrium, the photon distribution function has the Bose-Einstein distribution

$$f_{eq}(\eta, x^i, \vec{q}) = \frac{g}{(2\pi)^3} \frac{1}{e^{q/T} - 1}$$

(since there is no conservation law for the photon number, the chemical potential vanishes, $\mu = 0$, in equilibrium). Here $g = 2$ is the number of internal (two spin) degrees of freedom.

- In the background universe, thermal equilibrium for photons is a good enough approximation, and the temperature T is homogeneous. Thus

$$\bar{f}(\eta, x^i, \vec{q}) = \underline{\bar{f}(\eta, q) = \frac{g}{(2\pi)^3} \frac{1}{e^{q/T(\eta)} - 1}} \qquad (1)$$

- We write the perturbed distribution function as

$$\underline{f(\eta, x^i, q, \hat{n}) = \frac{g/(2\pi)^3}{\exp\left\{ \frac{q}{T(\eta)[1 + \Theta(\eta, x^i, q, \hat{n})]} \right\} - 1}} \qquad (2)$$

(completely general, Θ can be expressed in terms of f).

- Now assume $\underline{\Theta = \Theta(\eta, x^i, \hat{n})}$, independent of the photon energy q.

This assumption (or approximation) will be justified later. (Initially, at early times, photons are in equilibrium and the assumption holds. We will derive evolution equations for Θ, and they turn out not to contain q, so no q-dependence can develop. This actually depends on us using the approximation that when photons scatter on electrons, the photon energy does not change in the electron rest frame, which is good for $q \ll m_e$.)

- In local thermal equilibrium we would have local temperature $T(\eta, x^i) = T(\eta)[1 + \Theta(\eta, x^i)]$. Thus the assumption is that the deviation from this is just a direction dependence (anisotropy), not a distortion away from the black-body spectrum.

- $\Theta(\eta, x^i; \hat{n})$ is called the <u>brightness function</u>

An observer located at η, x^i will observe radiation coming towards him with a black-body spectrum, whose temperature depends on the direction $\hat{x} = -\hat{n}$ he is looking at.

$$T(-\hat{n}) = T(\eta) \cdot [1 + \Theta(\eta, x^i; \hat{n})]$$

- Since Θ is small (a 1st order perturbation) we can express the perturbation in the distribution function $f = \bar{f} + \delta f \equiv \bar{f} + f^{(1)}$ as

$$f^{(1)}(\eta, x^i; q, \hat{n}) \equiv \delta f = \frac{\partial \bar{f}}{\partial T} \cdot T(\eta) \Theta(\eta, x^i; \hat{n})$$

(since the perturbation is just the perturbation of T in $\bar{f}$ by $T\Theta$).

Since $\bar{f}$ depends on q and T only through the ratio q/T, we have

$$T \frac{\partial \bar{f}}{\partial T} = -q \frac{\partial \bar{f}}{\partial q} \quad \Rightarrow \quad \underline{f^{(1)} = -q \frac{\partial \bar{f}}{\partial q} \Theta} \qquad (3)$$

- Return now to the collisionless Boltzmann equation for photons,

$$\underline{\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} n^i + q \frac{\partial f}{\partial q} \left(-\mathcal{H} - \tfrac{1}{2} h'_{ij} n^i n^j - h'_{0i} n^i + \tfrac{1}{2} h_{00,i} n^i \right) = 0} \qquad (4)$$

The 0th order (background) part is

$$\frac{d\bar{f}}{d\eta} = \frac{\partial \bar{f}}{\partial \eta} - \mathcal{H} q \frac{\partial \bar{f}}{\partial q} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \bar{f}}{\partial \eta} = \mathcal{H} q \frac{\partial \bar{f}}{\partial q}} \qquad (5)$$

Since $\frac{\partial \bar{f}}{\partial \eta} = \frac{\partial \bar{f}}{\partial T} \frac{dT}{d\eta} = -\frac{q}{T} \frac{\partial \bar{f}}{\partial q} \frac{dT}{d\eta}$, the 0th order equation gives

$$\frac{1}{T} \frac{dT}{d\eta} = -\mathcal{H} \quad \Rightarrow \quad \frac{dT}{T} = -\mathcal{H} d\eta = -\frac{da}{a} \quad \Rightarrow \quad \underline{T \propto \frac{1}{a}}$$

.. Subtracting (5) from (4) we get the collisionless Boltzmann eq for perturbations

$$\frac{df^{(1)}}{d\eta} = \frac{\partial f^{(1)}}{\partial \eta} + \frac{\partial f^{(1)}}{\partial x^i} n^i - \mathcal{H} q \frac{\partial f^{(1)}}{\partial q} + q \frac{\partial \bar{f}}{\partial q}\left(-\tfrac{1}{2} h'_{ij} n^i n^j - h'_{0i} n^i + \tfrac{1}{2} h_{00,i} n^i\right) = 0 \qquad (6)$$

Using $f^{(1)} = -q \frac{\partial \bar{f}}{\partial q}\Theta$ we have

$$\frac{\partial f^{(1)}}{\partial \eta} = -q \frac{\partial^2 \bar{f}}{\partial q \partial \eta}\Theta - q \frac{\partial \bar{f}}{\partial q}\frac{\partial \Theta}{\partial \eta} \quad ; \quad \frac{\partial f^{(1)}}{\partial x^i} = -q \frac{\partial \bar{f}}{\partial q}\frac{\partial \Theta}{\partial x^i}$$

and $\frac{\partial f^{(1)}}{\partial q} = -\Theta \frac{\partial}{\partial q}\underbrace{\left(q \frac{\partial \bar{f}}{\partial q}\right)}_{\frac{1}{\mathcal{H}}\frac{\partial \bar{f}}{\partial \eta} \text{ from } (5)} = -\frac{\Theta}{\mathcal{H}}\frac{\partial^2 \bar{f}}{\partial q \partial \eta}$ which cancels the 1st term,

and we get

$$\frac{df^{(1)}}{d\eta} = -q \frac{\partial \bar{f}}{\partial q}\left[\frac{\partial \Theta}{\partial \eta} + \frac{\partial \Theta}{\partial x^i} n^i + \left(\tfrac{1}{2} h'_{ij} n^i n^j + h'_{0i} n^i - \tfrac{1}{2} h_{00,i} n^i\right)\right] = 0 \qquad (7)$$

Thus we have got the <u>Collisionless Brightness Equation</u>

$$\boxed{\frac{\partial \Theta}{\partial \eta} = \underbrace{-n^i \frac{\partial \Theta}{\partial x^i}}_{\text{free streaming}} \underbrace{- \tfrac{1}{2} h'_{ij} n^i n^j - h'_{0i} n^i + \tfrac{1}{2} h_{00,i} n^i}_{\text{redshift}}} \qquad (8)$$