F3. Photon Redshift and Free Streaming

- For photons, $E = \sqrt{q^2+m^2} = q \equiv |\vec{q}| = \sqrt{\delta_{ij}q^i q^j}$.

  Always, $q^i = q n^i$, $\delta_{ij} n^i n^j = 1$.

  $\hat{n} = n^i = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ gives the direction of the photon momentum in the observer ($x^i = \text{const}$) frame (the local orthonormal frame).

- We now develop the 1^{st} order collisionless Boltzmann equation for photons,

$$\frac{\partial f}{\partial \eta} = -\frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta} - \frac{\partial f}{\partial q}\frac{dq}{d\eta} \qquad (1)$$

- The photon energy q observed by the local observer can be written as

$$q \equiv -\underline{u}\cdot\underline{p} = -u_\mu p^\mu \qquad (2)$$

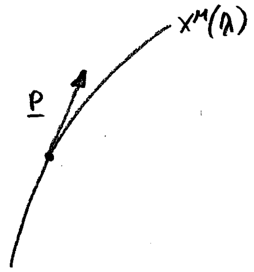
  where $\underline{u} = \underline{e}_{\hat{0}}$ is the 4-velocity of the observer

  and $\underline{p}$ is the photon 4-momentum.

- The contravariant components of the photon 4-momentum are

$$p^\mu = \frac{dx^\mu}{d\lambda} = \left(\frac{d\eta}{d\lambda}, \frac{dx^i}{d\lambda}\right)$$

  where λ is the affine parameter along photon world line



  They develop according to the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = \frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}p^\nu p^\rho = 0 \qquad (3)$$

- The components of the observer 4-velocity are

$$u^\mu = (\underline{e}_{\hat{0}})^\mu = a^{-1}(1+\tfrac{1}{2}h_{00}, 0, 0, 0)$$

$$u_\mu = g_{\mu\nu}u^\nu = a^2(\eta_{\mu 0}+h_{\mu 0})u^0 = a(-1+\tfrac{1}{2}h_{00}, h_{0i}) = -a(1-\tfrac{1}{2}h_{00}, -h_{0i}) \qquad (4)$$

- We can relate the photon 4-momentum components in the crd. frame, $p^\mu = (p^0, p^i)$ to those in the observer frame $p^{\hat{\mu}} = q^\mu = (q, q^i)$ by

$$p^\mu = X^\mu_{\hat{\nu}} q^\nu$$

$$\Rightarrow \quad p^0 = \bar{a}^1(1+\tfrac{1}{2}h_{00})q + \bar{a}^1 h_{0i} q^i = \bar{a}^1 q\,(1+\tfrac{1}{2}h_{00} + h_{0i}n^i)$$

$$p^i = \bar{a}^1(\delta_{ij} - \tfrac{1}{2}h_{ij})q^j = \bar{a}^1 q\,(n^i - \tfrac{1}{2}h_{ij}n^j) \qquad (5)$$

From this we actually only need below that to 0th order

$$p^0 = \bar{a}^1 q\,, \quad p^i = \bar{a}^1 q^i \quad \Rightarrow \quad \underline{\frac{p^i}{p^0} = n^i} \quad \text{to } 0^{th} \text{ order}$$

and that from

$$\underline{p}\cdot\underline{p} = g_{\mu\nu}p^\mu p^\nu = \underbrace{\eta_{\mu\nu}p^\mu p^\nu}_{\delta_{ij}p^i p^j - (p^0)^2} + h_{\mu\nu}p^\mu p^\nu = 0$$

$$\Rightarrow \quad \underline{\frac{\delta_{ij}p^i p^j}{(p^0)^2}} = 1 - h_{00} - 2h_{0i}\frac{p^i}{p^0} - h_{ij}\frac{p^i p^j}{(p^0)^2} = \underline{1 - h_{00} - 2h_{0i}n^i - h_{ij}n^i n^j} \qquad (6)$$

(to 1^{st} order)

- Since $\frac{\partial f}{\partial x^i}$ is a perturbation, we can approximate $\frac{dx^i}{d\eta}$ in Eq. (1) by

$$\underline{\frac{dx^i}{d\eta}} = \frac{p^i}{p^0} \underline{\simeq n^i \quad (\text{to } 0^{th} \text{ order})}$$

and write (1) as

$$\underline{\frac{\partial f}{\partial \eta} = -\frac{\partial f}{\partial x^i}n^i - q\frac{\partial f}{\partial q}\cdot\frac{1}{q}\frac{dq}{d\eta}} \qquad (7)$$

- We then proceed to evaluate the photon redshift, $\frac{1}{q}\frac{dq}{d\eta}$. Evaluate first $\frac{dq}{d\eta}$

- $$\frac{dq}{dy} = -\frac{d}{dy}(u_\mu p^\mu) = -\frac{du_\mu}{dy}p^\mu - u_\mu\frac{dp^\mu}{dy}$$

- Here the observer u_μ is defined everywhere by the crd.system and the metric and we can write
$$\frac{du_\mu}{dy} = \frac{\partial u_\mu}{\partial y} + \frac{\partial u_\mu}{\partial x^i}\frac{dx^i}{dy} = \frac{\partial u_\mu}{\partial y} + \frac{\partial u_\mu}{\partial x^i}\frac{p^i}{p^0}$$

Using u_μ from (4),

$$\frac{\partial u_0}{\partial y} = -a'(1-\tfrac{1}{2}h_{00}) + \tfrac{1}{2}ah_{00}' \qquad \frac{\partial u_0}{\partial x^i} = \tfrac{1}{2}ah_{00,i}$$

$$\frac{\partial u_i}{\partial y} = a'h_{0i} + ah_{0i}' \qquad \frac{\partial u_j}{\partial x^i} = ah_{0j,i}$$

$$\therefore \quad -\frac{du^\mu}{dy}p^\mu = \dots \quad \text{(exercise)}$$

- For $-u_\mu \frac{dp^\mu}{dy} = -u_\mu\frac{d\lambda}{dy}\frac{dp^\mu}{d\lambda} = -u_\mu\frac{1}{p^0}\frac{dp^\mu}{d\lambda}$ we need the geodesic equation (3)

The Christoffel symbols for a general $h_{\mu\nu}$ perturbation are

$$\Gamma^0_{00} = \mathcal{H} - \tfrac{1}{2}h_{00}' \qquad \Gamma^0_{0i} = \mathcal{H}h_{0i} - \tfrac{1}{2}h_{00,i}$$

$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} + \mathcal{H}\delta_{ij}h_{00} + \mathcal{H}h_{ij} - \tfrac{1}{2}(h_{0i,j}+h_{0j,i}) + \tfrac{1}{2}h_{ij}'$$

$$\Gamma^i_{00} = \mathcal{H}h_{0i} + h_{0i}' - \tfrac{1}{2}h_{00,i} \qquad \Gamma^i_{0j} = \mathcal{H}\delta_{ij} - \tfrac{1}{2}(h_{0j,i} - h_{0i,j}) + \tfrac{1}{2}h_{ij}'$$

$$\Gamma^i_{jk} = -\mathcal{H}\delta_{jk}h_{0i} + \tfrac{1}{2}h_{ij,k} + \tfrac{1}{2}h_{ik,j} - \tfrac{1}{2}h_{jk,i}$$

- The result is (exercise)

$$\frac{dq}{dy} = p^0\left(-a' + \tfrac{1}{2}a'h_{00} + \tfrac{1}{2}ah_{00,i}n^i + a'h_{0i}n^i - ah_{0i}'n^i - \tfrac{1}{2}ah_{ij}'n^in^j\right)$$

We also need $q = -g_{\mu\nu}u^\mu p^\nu = -a^2(\eta_{\mu\nu}+h_{\mu\nu})u^\mu p^\nu = \dots = ap^0(1-\tfrac{1}{2}h_{00} - h_{0i}n^i)$

to arrive at the final result (exercise)

$$\boxed{\frac{1}{q}\frac{dq}{dy} = -\mathcal{H} - \tfrac{1}{2}h_{ij}'n^in^j - h_{0i}'n^i + \tfrac{1}{2}h_{00,i}n^i}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x^i}n^i - q\frac{\partial f}{\partial q}\left(-\mathcal{H} - \tfrac{1}{2}h_{ij}'n^in^j - h_{0i}'n^i + \tfrac{1}{2}h_{00,i}n^i\right)$$