F2. Energy Tensor in the Local Orthonormal Frame

- The energy tensor $T^{\hat{\mu}\hat{\nu}}$ in the local orthonormal frame is given in terms of the distribution function as

$$T^{\hat{\mu}\hat{\nu}}(\eta, x^i) = \int \frac{d^3q^i}{E} f(\eta, x^i, \vec{q})\, q^\mu q^\nu \tag{1}$$

where $q^\mu \equiv (E, q^i)$, the 4-momentum in the local orthonormal frame.

Thus we have

$$\begin{aligned} T^{\hat{0}\hat{0}} &= \int E f\, d^3q && \text{energy density} \\ T^{\hat{0}\hat{i}} &= \int q^i f\, d^3q && \text{momentum density} = \text{energy flux} \\ T^{\hat{i}\hat{j}} &= \int \frac{q^i q^j}{E} f\, d^3q && \text{momentum flux} \end{aligned} \tag{2}$$

for the quantities observed by a comoving (with the rad. system) observer.

- In the orthonormal frame $T^{\hat{\mu}}_{\hat{\nu}} = \eta_{\nu\beta} T^{\hat{\mu}\beta}$, so

$$\begin{aligned} T_{\hat{0}\hat{0}} &= -T^{\hat{0}}_{\hat{0}} = T^{\hat{0}\hat{0}} \\ T_{\hat{0}\hat{i}} &= T^{\hat{i}}_{\hat{0}} = -T^{\hat{0}\hat{i}} \\ T_{\hat{i}\hat{j}} &= T^{\hat{i}}_{\hat{j}} = T^{\hat{i}\hat{j}} \end{aligned} \tag{3}$$

- To make contact w the earlier development of perturbation theory, we must transform

$$T^\mu_\nu = \begin{bmatrix} -\bar{\rho} - \delta\rho & (\bar{\rho}+\bar{p})(v_i + h_{0i}) \\ -(\bar{\rho}+\bar{p})v_i & (\bar{p}+\delta p)\delta_{ij} + \Sigma_{ij} \end{bmatrix} \tag{4}$$

into the orthonormal frame.

- Using $g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$ and $g^{\mu\nu} = a^{-2}(\eta^{\mu\nu} - h^{\mu\nu})$

we have for the contravariant and covariant components in the crd frame:

$$T^{\mu\nu} = a^{-2}\begin{bmatrix} \bar{\rho}+\delta\rho+\bar{\rho}h_{00} & (\bar{\rho}+\bar{p})v_i + \bar{p}h_{0i} \\ (\bar{\rho}+\bar{p})v_i + \bar{p}h_{0i} & (\bar{p}+\delta p)\delta_{ij} + \Sigma_{ij} - \bar{p}h_{ij} \end{bmatrix} \tag{5}$$

$$T_{\mu\nu} = a^{2}\begin{bmatrix} \bar{\rho}+\delta\rho-\bar{\rho}h_{00} & -(\bar{\rho}+\bar{p})v_i - \bar{\rho}h_{0i} \\ -(\bar{\rho}+\bar{p})v_i - \bar{\rho}h_{0i} & (\bar{p}+\delta p)\delta_{ij} + \Sigma_{ij} + \bar{p}h_{ij} \end{bmatrix} \tag{6}$$

- How to <u>construct the orthonormal basis</u> $\{\underline{e}_{\hat{\mu}}\}$ that corresponds to a given coordinate basis $\{\underline{e}_\mu \equiv \partial_\mu\}$, i.e. the one the corresponds to an observer at fixed crd's x^i? For the case of an orthogonal crd. system this is trivial, just normalize the $\underline{e}_\mu$. But we want to do the general case where $h_{\mu\nu}$ is not diagonal. The dual basis $\{\underline{e}^\mu \equiv \underline{dx}^\mu\}$ comes to our aid, since

$$\underline{e}_\mu \cdot \underline{e}^\nu = \delta_\mu^\nu \tag{7}$$

From GR we have that the contravariant and covariant components of the crd. basis and its dual basis are

$$(\underline{e}_\mu)^\alpha = \delta_\mu^\alpha \qquad (\underline{e}_\mu)_\alpha = g_{\alpha\mu}$$
$$(\underline{e}^\nu)_\alpha = \delta_\alpha^\nu \qquad (\underline{e}^\nu)^\alpha = g^{\alpha\nu} \tag{8}$$

and their inner products are

$$\underline{e}_\mu \cdot \underline{e}_\nu = (\underline{e}_\mu)_\alpha (\underline{e}_\nu)^\alpha = g_{\mu\nu}$$
$$\underline{e}^\mu \cdot \underline{e}^\nu = (\underline{e}^\mu)_\alpha (\underline{e}^\nu)^\alpha = g^{\mu\nu} \tag{9}$$

- From these, we want to construct a new set of basis vectors $\{\underline{e}_{\hat{\mu}}\}$, so that $\underline{e}_{\hat{\mu}} \cdot \underline{e}_{\hat{\nu}} = \eta_{\mu\nu}$ and $\underline{e}_{\hat{0}}$ is tangent to the $x^i = \text{const}$ line.

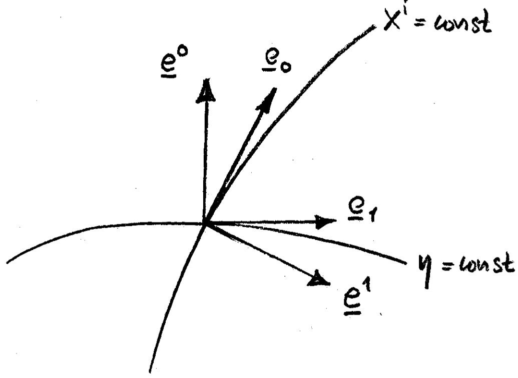

Figure: Coordinate basis vectors $\underline{e}_\mu$ and their dual basis vectors $\underline{e}^\mu$. Note that the figure is exaggerated; the deviation from orthogonality is just a 1st order perturbation.

- Since $\underline{e}_0$ is already tangent to $x^i = \text{const}$, it just has $\underline{e}_0 \cdot \underline{e}_0 = g_{00} = -a^2(1-h_{00})$ instead of -1, we get

$$\underline{\underline{e}_{\hat{0}} \equiv \frac{1}{\sqrt{|g_{00}|}} \underline{e}_0} = \frac{1}{a\sqrt{1-h_{00}}} \underline{e}_0 = \underline{a^{-1}\left(1+\tfrac{1}{2}h_{00}\right)\underline{e}_0} \qquad (10)$$

- The $\underline{e}^i$ are orthogonal to $\underline{e}_0$, but not to each other. However, if we define

$$\underline{\underline{e}_{\hat{\imath}} \equiv a\left(\delta_{ij} + \tfrac{1}{2}h_{ij}\right)\underline{e}^j} \qquad (11)$$

we get $\underline{\underline{e}_{\hat{\imath}} \cdot \underline{e}_{\hat{\jmath}}} = a\left(\delta_{ik} + \tfrac{1}{2}h_{ik}\right) a\left(\delta_{jl} + \tfrac{1}{2}h_{jl}\right) \underline{e}^k \cdot \underline{e}^l$

$$= \left(\delta_{ik} + \tfrac{1}{2}h_{ik}\right)\left(\delta_{jl} + \tfrac{1}{2}h_{jl}\right)\left(\delta_{kl} - h_{kl}\right) = \delta_{ij} - h_{ij} + \tfrac{1}{2}h_{ij} + \tfrac{1}{2}h_{ij} = \underline{\delta_{ij}}$$

and $\underline{e}_{\hat{\imath}} \cdot \underline{e}_{\hat{0}} = 0$. Thus $\{\underline{e}_{\hat{\mu}}\}$ is an orthonormal basis, $\underline{\underline{e}_{\hat{\mu}} \cdot \underline{e}_{\hat{\nu}} = \eta_{\mu\nu}}$

- In terms of the crd. basis $\{\underline{e}_\mu\}$, $\underline{\underline{e}_{\hat{0}} = a^{-1}\left(1+\tfrac{1}{2}h_{00}\right)\underline{e}_0}$ and

$$\underline{\underline{e}_{\hat{\imath}}} = a\left(\delta_{ij} + \tfrac{1}{2}h_{ij}\right)\underline{e}^j = a\left(\delta_{ij} + \tfrac{1}{2}h_{ij}\right) g^{j\mu}\underline{e}_\mu = a\left(\delta_{ij} + \tfrac{1}{2}h_{ij}\right) a^{-2}\left(\eta^{j\mu} - h^{j\mu}\right)\underline{e}_\mu$$

$$= \ldots = \underline{a^{-1}\left[h_{0i}\underline{e}_0 + \left(\delta_{ij} - \tfrac{1}{2}h_{ij}\right)\underline{e}_j\right]} \qquad (12)$$

or $\underline{e}_{\hat{\mu}} = X^\nu_{\hat{\mu}} \underline{e}_\nu$ where

$$\boxed{\begin{array}{ll} X^0_{\hat{0}} = a^{-1}\left(1+\tfrac{1}{2}h_{00}\right) & X^0_{\hat{\imath}} = a^{-1}h_{0i} \\ X^i_{\hat{0}} = 0 & X^j_{\hat{\imath}} = a^{-1}\left(\delta_{ij} - \tfrac{1}{2}h_{ij}\right) \end{array}} \qquad (13)$$

is the transformation matrix between the two bases.

- We can now use $X^\nu_{\hat\mu}$ to transform the covariant components of any tensor from the crd. basis to this orthonormal basis. For the energy tensor we have

$$T_{\hat\mu\hat\nu} = X^\rho_{\hat\mu} X^\sigma_{\hat\nu} T_{\rho\sigma}$$

which, using (5) & (12), gives (exercise)

$$\boxed{\begin{aligned} T_{\hat0\hat0} &= T^{\hat0\hat0} &&= \bar\rho + \delta\rho \\ T_{\hat0\hat\imath} &= T_{\hat\imath\hat0} &&= -(\bar\rho+\bar p)v_i \\ T^{\hat0\hat\imath} &= T^{\hat\imath\hat0} &&= +(\bar\rho+\bar p)v_i \\ T_{\hat\imath\hat\jmath} &= T^{\hat\imath\hat\jmath} &&= (\bar p+\delta p)\delta_{ij} + \Sigma_{ij} \end{aligned}} \qquad (14)$$

- Thus we get the connection between energy tensor perturbations and the distribution function:(*

$$\begin{aligned} \rho &= \bar\rho+\delta\rho &&= \int E f\, d^3q \\ p &= \bar p+\delta p = \tfrac{1}{3}\delta_{ij}T^{\hat\imath\hat\jmath} &&= \int \frac{\delta_{ij}q^iq^j}{3E} f\, d^3q \\ \Sigma_{ij} &= T^{\hat\imath\hat\jmath} - \delta_{ij}p &&= \int \left(q^iq^j - \tfrac{1}{3}\delta^{ij}q^2\right)\frac{f\,d^3q}{E} \\ (\rho+p)v_i &= (\bar\rho+\bar p)v_i &&= \int q^i f\, d^3q \end{aligned} \qquad (15)$$

$$q \equiv \sqrt{\delta_{ij}q^iq^j} = |\vec q|$$

*) Note that these results hold only in 1^{st} order perturbation theory. For example, we have assumed that the fluid velocity $\vec v$ is small, i.e., nonrelativistic. In general, in GR, the observed energy density $T^{\hat0\hat0}$ is not equal to the energy density ρ in the fluid rest frame, due to two reasons: 1) the kinetic energy due to the fluid velocity $\vec v$ 2) Lorentz contraction of the volume element. Both of these are $\mathcal{O}(v^2)$ effects, and thus get ignored in 1^{st} order perturbation theory. (Particle velocities $\vec q/E$ may be relativistic in our treatment – and most certainly are for photons.)