F. COLLISIONLESS BOLTZMANN EQUATION

F1. Phase Space and the Distribution Function

- We have seen that treating the matter (energy) content of the universe as a fluid described by the energy tensor T^μ_ν does not lead to a complete set of equations, when the perfect fluid approximation is not valid. Also, this fluid description of the cosmic photon background cannot describe the CMB anisotropy (T^μ_ν describes multipoles up to $L=2$ only.)

- Before photon decoupling, baryons and photons behave as a single baryon-photon fluid, for which the perfect fluid approximation is valid.

- During photon decoupling the perfect fluid approximation breaks down, and we require a more detailed treatment of the matter and energy content of the universe.
  For the components (baryons, CDM, neutrinos, photons) which consist of particles this more detailed treatment is provided by the distribution function f in the 6-dimensional phase space. Each particle species i has its own distribution function f_i.

- The phase space is described by six variables:

  three coordinates x^i

  and their conjugate momenta p_i

- For our case of particles in a curved spacetime with some crd. system x^μ, we can take the coordinates to be the three space coordinates x^i. The corresponding conjugate momenta are then the covariant space components p_i of the particle 4-momentum $\underline{p}$ in the coordinate basis.

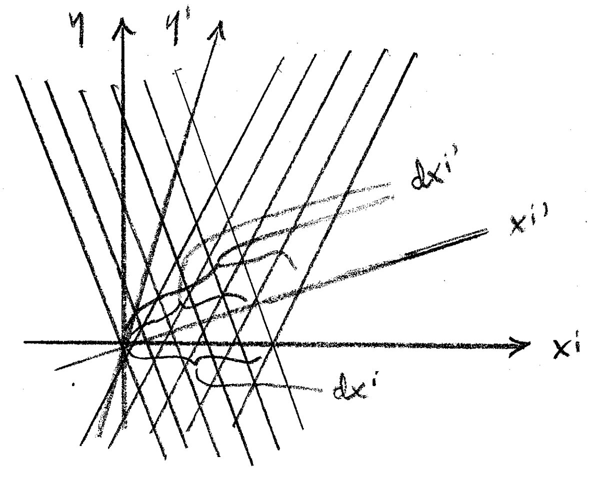
- We define the distribution function f of a particle species as the 6-dimensional number density of these particles in phase space. Thus the number dN of particles at space coordinates x^i, within $dx^1 dx^2 dx^3$, and with 4-momentum space components p_i, within $dp_1 dp_2 dp_3$, at time η, is given by

$$dN = f(\eta, x^i, p_i) \underbrace{dx^1 dx^2 dx^3 dp_1 dp_2 dp_3}_{\text{phase space volume element}} \qquad (1)$$

Thus f is a function of time also, but the time coordinate (here η) appears in a different role than the space coordinates; since f is defined as a density wrt the space coordinates. So $f(\eta, x^i, p_i)$ is a function in the 6-dim phase space $\{x^i, p_i\}$ which evolves in time η. The time component p_0 of the particle 4-momentum does not appear as a variable, since for a given particle species (w given mass m), p_0 is determined by the p_i (and the metric), and is thus not an independent variable.

- It turns out that <u>the distribution function f</u>, defined this way, <u>is independent of the choice of crd. systems!</u> We skip the proof of this important fact. (In CMB Physics 2004 it was done for the case of Minkowski space.) But note what is meant by it:

Consider a set of $\overset{dN}{\text{particles}}$, all w momentum p_i (within d^3p_i) and location x^i (within d^3x^i) in one crd. system. These same particles have a momentum $p_{i'}$ (within $d^3p_{i'}$) and location $x^{i'}$ (within $d^3x^{i'}$) in another crd. system. Note that the volume $d^3x^{i'}$ they occupy in the new crd. system depends on $p_{i'}$ (their velocity) – d^3x^i cannot be transformed independently of p_i! (See Figure 1, which shows two sets of particles w different p_i.)



By definition the number of these particles is the same dN (since we consider a given set of particles) in both crd. systems. The claim is that the 6-dim volume elements are equal:

$$dx^{1'}dx^{2'}dx^{3'}dp_{1'}dp_{2'}dp_{3'} = dx^1dx^2dx^3dp_1dp_2dp_3 \qquad (2)$$

(This is the statement whose proof we skip.)

- From this follows that

$$\underbrace{\underbrace{f(y', x^{i'}, p_{i'})}_{} = \underbrace{f(y, x^i, p_i)}_{}}_{\text{these refer to the same set of particles at the same spacetime event.}} \qquad (3)$$

Thus f is a scalar quantity. For a given system, a given particle species, a given spacetime event P, and a given 4-momentum $\underline{p}$ (appropriate for this particle species), $f = f(P, \underline{p})$ is independent of the crd. system used to describe the system.

- We shall use a global crd. system $\{y, x^i\}$ in our discussion of the evolution of the distribution function. However, it is more convenient to use the 3-momentum q^i observed by a local observer instead of the covariant components p_i of the coordinate frame. For a given observer at a given spacetime event P we can always construct a locally orthonormal crd. system $\{x^{\hat\mu}\}$ so that $p_{\hat i} = p^{\hat i} = q^i$ at P. Thus, at P

$$dN = f(\hat t, x^{\hat i}, q^i)\, d^3x^{\hat i}\, d^3q^i = f(y, x^i, p_i)\, d^3x^i\, dp_i$$

Now $\{x^{\hat\mu}\}$ do not remain orthonormal outside of P. On the other hand, in a given crd. system $\{x^\mu\}$, we can define for every spacetime event a comoving observer, one whose space crd's x^i remain constant, and obtain the 3-momentum components q^i in his local orthonormal frame. We discuss the distribution function f in terms of these variables, $f(y, x^i, q^i)$. It is the same quantity and thus has the same values as before

$$f(y, x^i, q^i) = f(t, x^{\hat i}, p^{\hat i}) = f(y, x^i, p_i),$$

but in terms of these variables

$$dN \neq f\, d^3x^i\, d^3q^i$$

since the q^i are not the conjugate momenta of x^i. Instead we have, since

$$dV = d^3x^{\hat i} = \sqrt{\det(g_{jk})}\, d^3x^i$$

that

$$\underline{dN = f\, d^3x^{\hat i}\, d^3q^i = \sqrt{\det(g_{jk})} \cdot f\, d^3x^i\, dq^i} \qquad (4)$$

- From Statistical Physics, we have the important result:

<u>Liouville Theorem:</u> In the absence of collisions between particles, f stays constant along any particle trajectory in phase space,

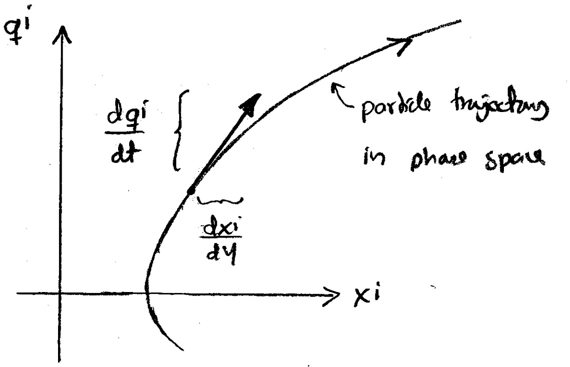
$$\frac{df}{dy} = 0 \quad (5)$$

- Since f has 7 variables, $f = f(y, x^i, q^i)$, we can write this in terms of partial derivatives,

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dy} + \frac{\partial f}{\partial q^i}\frac{dq^i}{dy} = 0 \quad (6)$$

This is called the Collisionless Boltzmann Equation

where $\frac{dx^i}{dy}$ describes the motion of the particle (w momentum q^i) and $\frac{dq^i}{dy}$ how its momentum changes in time.



- Instead of the three components of the locally observed momentum q^i, we could also use the variables

$$E \equiv \sqrt{\delta_{ij} q^i q^j + m^2} \quad (7)$$

the particle energy observed by the local observer, and the angles ϑ, φ specifying the direction

$$\hat{n} \equiv \frac{\vec{q}}{|\vec{q}|} \quad (8)$$

where the particle is going. So we can write (6) also as

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dy} + \frac{\partial f}{\partial E}\frac{dE}{dy} + \underbrace{\frac{\partial f}{\partial \vartheta}\frac{d\vartheta}{dy} + \frac{\partial f}{\partial \varphi}\frac{d\varphi}{dy}}_{\text{"}\frac{\partial f}{\partial \hat{n}}\frac{d\hat{n}}{dy}\text{"}} = 0 \quad (9)$$

for $f(y, x^i, E, \hat{n})$.

(We could also use $q \equiv |\vec{q}|$ instead of E: $f(y, x^i, q, \hat{n})$.)

- In the background universe the distribution function $\bar{f}$ must be homogeneous and isotropic,

$$\bar{f}(\eta, x^i, E, \hat{n}) = \bar{f}(\eta, E) \tag{10}$$

$\Rightarrow$ the partial derivatives $\frac{\partial f}{\partial x^i}$ and $\frac{\partial f}{\partial \hat{n}}$ are perturbations.

- Since the background universe is isotropic, a free particle does not change direction in the bg universe.

$\Rightarrow$ $\frac{d\hat{n}}{d\eta}$ is a perturbation

$\therefore$ We have

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \underbrace{\frac{\partial f}{\partial x^i}}_{1^{st}\text{ order}} \frac{dx^i}{d\eta} + \frac{\partial f}{\partial E}\frac{dE}{d\eta} + \underbrace{\overbrace{\frac{\partial f}{\partial \hat{n}}}^{1^{st}\text{ order}} \cdot \overbrace{\frac{\partial \hat{n}}{d\eta}}^{1^{st}\text{ order}}}_{2^{nd}\text{ order}} = 0$$

Thus we have to 0^{th} order

$$\frac{d\bar{f}}{d\eta} = \frac{\partial \bar{f}}{\partial \eta} + \frac{\partial \bar{f}}{\partial E}\frac{dE}{d\eta} = 0 \tag{11}$$

and to 1^{st} order

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial f}{\partial E}\frac{dE}{d\eta} = 0 \tag{12}$$

We can write this as

$$\frac{\partial f}{\partial \eta} = \underbrace{-\frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta}}_{\text{free streaming}} \underbrace{-\frac{\partial f}{\partial E}\frac{dE}{d\eta}}_{\text{redshift}} \tag{13}$$

This result, the collisionless Boltzmann equation to 1^{st} order, expresses how the value of the distribution function at x^i, q^i changes in time due to two effects:

1) Free streaming: Due to their velocities $\frac{dx^i}{d\eta}$, particles move out of the volume element d^3x^i (and thus from the phase space element $d^3x^i d^3q^i$), and other particles move in.

2) Redshift: The energy (and thus the momentum) of the particle changes due to gravity.