Appendix: Detector Beam

Intensity

- The treatment of the detector beam is more complicated for a polarization-sensitive detector, so we discuss here first the case of a detector which detects just the intensity of the radiation, i.e., the Stokes parameter I. The starting point is thus Eq. (12) of §P8,

$$\underline{P = A\int_{\Delta\Omega} d\Omega \int_0^\infty d\nu F(\nu) B_\nu(\nu,\hat{n})} \tag{1}$$

with a very crude beam description: a "top-hat" beam, where all photons with momentum direction $\hat{n} \in \Delta\Omega$ are detected with the same effective aperture A, and those that fall outside $\Delta\Omega$ have no effect on the detector.

- In the following we assume that the beam shape (of a given detector) does not vary as a function of frequency ν

- We can improve Eq. (1) by including the dependence of the detector response on the photon direction:

$$P \equiv A\int_{4\pi} d\Omega\, b(\hat{n}) \int_0^\infty d\nu F(\nu) B_\nu(\nu,-\hat{n}) \tag{2}$$

Here we wrote $-\hat{n}$ in B_ν which describes the photon momentum distribution, since we now take $\hat{n}$ to refer to the direction we are looking at, and photons coming from that direction of course have momentum direction $-\hat{n}$.

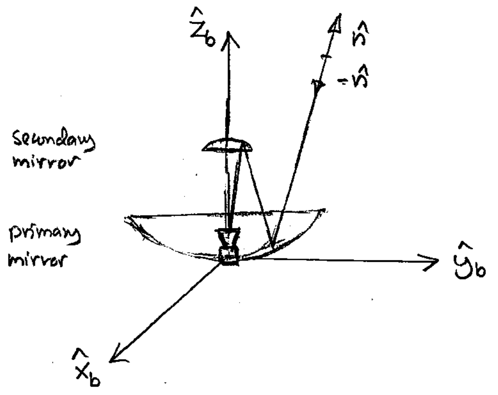
Here $b(\hat{n})$ describes the detector beam. In Eq. (2) the value of the constant A depends on the normalization of $b(\hat{n})$. We can either normalize $b(\hat{n})$ so that $b(\hat{n}_0) = 1$, where $\hat{n}_0$ is the direction of the beam center, which typically corresponds to the maximum of $b(\hat{n})$. In this case A is the collecting area for photons coming from that direction. Then

$$\Delta\Omega = \int_{4\pi} d\Omega\, b(\hat{n}) \tag{3}$$

is the (effective) beam solid angle. Or we can normalize $\int_{4\pi} d\Omega\, b(\hat{n}) = 1$, (4)

in which case A becomes something else.

- The beam $b(\hat{n})$ in Eq. (3) depends on how the instrument is oriented, i.e., at which direction $\hat{n}_0$ it is pointed at. In the following $\hat{n}$ (a direction from which we are observing photons) and $\hat{n}_0$ (a direction we are pointing the instrument at) are direction vectors that exist independent of crd. systems, and $\vartheta, \varphi, \vartheta_0, \varphi_0$; $\vartheta', \varphi', \vartheta_0', \varphi_0'$ are their spherical coordinates in variously oriented crd. systems.



Introduce the beam crd. system x_b, y_b, z_b and the corresponding ϑ^b, φ^b, which is attached to the detector so that $\hat{z}_b = \hat{n}_0$, i.e. the beam center is at $\vartheta^b = 0$.

Then we have another, "astronomical", crd. system (e.g., equatorial, ecliptic, galactic) that we call the sky crd. system. x, y, z and ϑ, φ.

In the beam crd's the beam

$$\underline{b(\vartheta^b, \varphi^b) = \sum_{\ell m} b^{(b)}_{\ell m} Y^m_\ell(\vartheta^b, \varphi^b)} \tag{4}$$

is a fixed function of ϑ_b and φ_b. If the detector is oriented so that $(\hat{x}_b, \hat{y}_b, \hat{z}_b)$ are aligned with $(\hat{x}, \hat{y}, \hat{z})$ – i.e., the detector is looking at the "North Pole" (NP) of the sky crd system <u>and</u> also $\hat{y}_b$ and $\hat{z}_b$ axis are aligned with $\hat{y}$ and $\hat{z}$ – then $\vartheta^b = \vartheta$, $\varphi^b = \varphi$ and

$$b(\vartheta, \varphi) = \sum_{\ell m} b^{(b)}_{\ell m} Y^m_\ell(\vartheta, \varphi)$$

However, if the detector is oriented differently; looking at some other direction $\hat{n}_0 \neq \hat{z}$, or just rotated around $\hat{z} = \hat{n}_0$ so that $\hat{x}_b \neq \hat{x}$, $\hat{y}_b \neq \hat{y}$ (the latter matters only if the beam is not circularly symmetric, i.e., depends on φ_0), then

$$b(\vartheta, \varphi) = \sum_{\ell m} b_{\ell m} Y^m_\ell(\vartheta, \varphi)$$

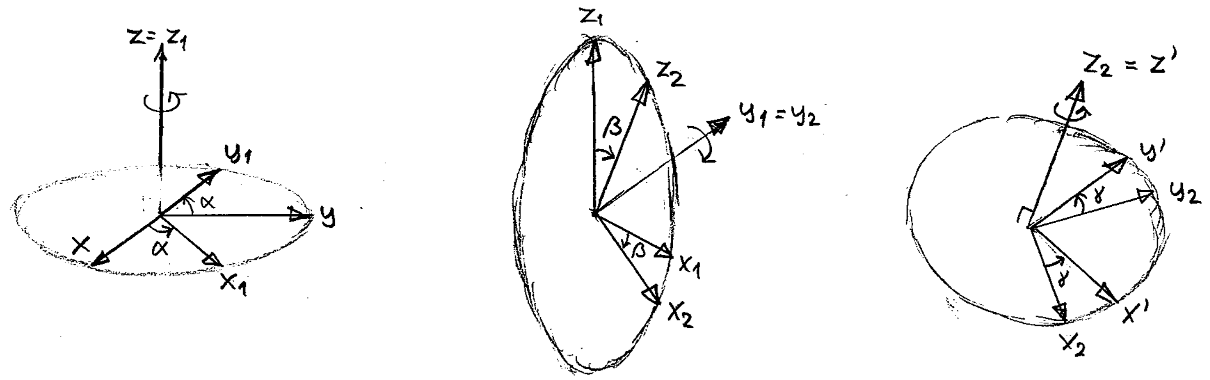
is some different function of the sky ϑ, φ. In fact

$$\underline{b_{\ell m} = \sum_{m'} D^\ell_{m m'}(\alpha, \beta, \gamma)\, b^{(b)}_{\ell m'}} \tag{5}$$

where α, β, γ are the Euler angles corresponding to rotating the crd. system from the sky crd's ϑ, φ to the beam crd's ϑ^b, φ^b.

Euler Angles

- The Euler angles α, β, γ of a rotation of crd. system $x, y, z \to x', y', z'$ are defined as follows:



The rotation can be performed in three steps:

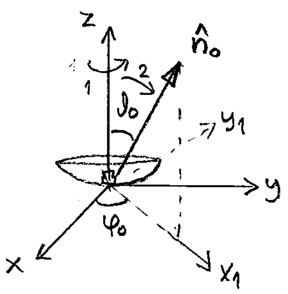
1) Rotate the crd system around the original z axis by an angle α $(0 \le \alpha < 2\pi)$
Call the new crd's x_1, y_1, z_1

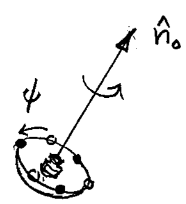
2) Rotate the crd. system around the new y_1 axis by an angle β $(0 \le \beta < \pi)$
Call the new crd's x_2, y_2, z_2

3) Rotate the crd. system around the new z_2 axis by an angle γ $(0 \le \gamma < 2\pi)$
Call the new crd's x', y', z'.

Rotating the Detector and Its Beam

- The rotation of the detector from the "reference" orientation $\hat{x}_D = \hat{x}$, $\hat{y}_D = \hat{y}$, $\hat{z}_D = \hat{z}$ to its actual orientation $\hat{x}_D, \hat{y}_D, \hat{z}_D = \hat{n}_0$ when it is pointing in the direction $\hat{n}_0$ corresponds to a rotation of the crd. system $x, y, z \to x_D, y_D, z_D$ (the beam crd's rotate together w the detector). Suppose $\hat{n}_0$ has sky crd's ϑ_0, φ_0.





1) Rotate the detector around the z axis by angle φ_0, to get new axes x_1 and y_1. The detector is still pointing at NP = the direction $\hat{z}$. $\qquad \varphi_0 = \alpha$

2) Rotate the detector around the y_1 axis by angle ϑ_0. This causes the detector to point in the direction $\hat{n}_0$ (the $z_2 = z'$ axis). $\qquad \vartheta_0 = \beta$.

3) We may still need to rotate the detector around $\hat{n}_0$ by some angle ψ to arrive at the actual detector orientation we want to consider. The pointing stays in the direction $\hat{n}_0$. $\qquad \psi = \gamma$

- Thus we have that the beam shape $b(\vartheta,\varphi)$ in the sky coordinates, when the detector is pointing at $\hat{n}_o = (\vartheta_o, \varphi_o)$ and has orientation ψ around the beam axis $\hat{n}_o$, is

$$\boxed{\begin{aligned} b(\vartheta,\varphi) &= \sum_{\ell m m'} D^{\ell}_{mm'}(\varphi_o,\vartheta_o,\psi)\, b^{(b)}_{\ell m'}\, Y^m_\ell(\vartheta,\varphi) \\ &\equiv B(\hat{n},\hat{n}_o,\psi) \end{aligned}} \qquad (6)$$

Where we defined $B(\hat{n},\hat{n}_o,\psi)$ so that it gives the beam as a function of sky direction $\hat{n}$ for a detector (beam) orientation $\hat{n}_o, \psi$. The full function $B(\hat{n},\hat{n}_o,\psi)$ is independent of what sky crd's we use for $\hat{n}$ and $\hat{n}_o$; except that the specification of the orientation angle ψ depends on the relation between the beam and sky crd. system.

- Referring back to Eq. (2), we have that the power received by the detector when it is pointing at $\hat{n}_o$ with orientation ψ (around $\hat{n}_o$) is

$$\boxed{P(\hat{n}_o,\psi) = A\int_0^\infty d\nu\, F(\nu) \int_{4\pi} d\Omega\, B(\hat{n},\hat{n}_o,\psi)\, B_\nu(\nu,-\hat{n})} \qquad (7)$$

Here
$$B^{(bs)}_\nu(\hat{n}_o,\psi,\nu) \equiv \int_{4\pi} d\Omega\, B(\hat{n},\hat{n}_o,\psi)\, B_\nu(\nu,-\hat{n}) \qquad (8)$$

is the convolution of the "sky" (the photon brightness as a function of direction) $B_\nu(\nu,-\hat{n})$ with the beam $B(\hat{n},\hat{n}_o,\psi)$.

- The beam in its own crd. system is given by $B(\hat{n},\hat{z},0) = b(\vartheta^b,\varphi^b)$ (9) where $\hat{n} = (\vartheta^b,\varphi^b)$.

Circularly Symmetric Beam

- If the beam is (or is approximated as) symmetric wrt rotation around the beam axis, i.e., independent of φ^b, $\quad b(\vartheta^b,\varphi^b) = b(\vartheta^b) \quad \Rightarrow \quad b^{(b)}_{\ell m} = 0$ for $m \neq 0$

$$\Rightarrow\ b(\vartheta^b,\varphi^b) = \sum_\ell b^{(b)}_{\ell 0}\, Y^0_\ell(\vartheta^b,\varphi^b) = \sum_\ell \sqrt{\frac{2\ell+1}{4\pi}}\, b^{(b)}_{\ell 0}\, P_\ell(\cos\vartheta^b) \qquad (10)$$

then $B(\hat{n},\hat{n}_o,\psi) = B(\hat{n},\hat{n}_o)$ is independent of the orientation ψ and we can define

$$B^{(bs)}_\nu(\hat{n}_o,\nu) \equiv \int_{4\pi} d\Omega\, B(\hat{n},\hat{n}_o)\, B_\nu(\nu,-\hat{n}), \qquad (11)$$

the beam-smoothed sky (at frequency ν).

- Convolving also with the frequency response,

$$\underline{B^{(\mathrm{los})}(\hat{n}_0)} \equiv \int_0^\infty d\nu \, F(\nu) \underline{B_\nu^{(\mathrm{los})}(\hat{n}_0, \nu)} \qquad (12)$$

we get how the sky looks (observed brightness as a function of observing direction $\hat{n}_0$) as observed through the detector.

Actually, this of course assumes an ideal detector, e.g., without any noise.

Asymmetric Beam and Polarization

- If the beam is asymmetric (depends on φ_b) the observed brightness at direction $\hat{n}_0$ depends on the detector orientation ψ. The result of a sky survey depends therefore, besides the detector properties, also on the scanning strategy, i.e., the distribution of different orientations ψ in which the detector looked at each direction $\hat{n}_0$. An asymmetric beam leads to a distortion of the image of the sky that is not just the simple smoothing produced by a symmetric beam, and the distortion will vary from place to place (as a function of $\hat{n}_0$) depending on the scanning strategy.
- As the scanning strategy and beam shape are known, one may try to correct for this distortion by solving for the undistorted sky from the set of observations

$$\{ P(\hat{n}_0, \psi) \}$$

with different $\hat{n}_0$ and ψ. This is called beam deconvolution.

- Also in the case of a polarization-sensitive detector the orientation ψ of the detector is important. The polarization angle ψ^b_{pol} to which the detector is sensitive to, is defined wrt the x_b and y_b axes of the beam crd system, which is attached to the detector. What polarization direction this corresponds to on the sky depends on the detector orientation $\hat{n}_0, \psi$.

- We define polarization directions on the sky wrt ϑ, φ crd. system

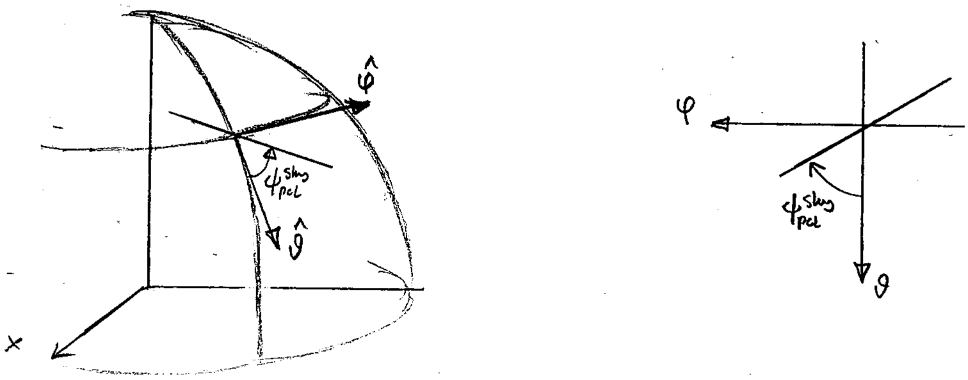


Fig. The celestial sphere ("sky") seen a) from "outside", b) from "inside", as we see it.

From the discussion on the bottom of p. DB-3 about rotating the detector to its observing position, we see that if $\psi = 0$: the directions of the beam axes are aligned w $\hat{\vartheta}$ and $\hat{\varphi}$,

i.e., $\hat{x}_b = \hat{\vartheta}$, $\hat{y}_b = \hat{\varphi}$ $\Rightarrow$ $\psi^{b}_{pol} = \psi^{sky}_{pol}$

where ψ^{sky}_{pol} is the linear polarization direction wrt the sky crd's ϑ, φ, the detector is sensitive to. If the detector is then rotated by an angle ψ (step 3), the polarisation sensitive direction rotates with it:

$$\underline{\psi^{sky}_{pol} = \psi^{b}_{pol} + \psi} \qquad (13)$$

Thus the power received by an ideal polarization-sensitive detector is thus

$$\boxed{P(\hat{n}_0, \psi) = A \int_0^\infty d\nu F(\nu) \int_{4\pi} d\Omega \, B(\hat{n}, \hat{n}_0, \psi) \cdot \frac{1}{2} \left[I_B(\nu, \hat{n}) + Q_B(\nu, \hat{n}) \cos 2(\psi^{b}_{pol} + \psi) + U_B(\nu, \hat{n}) \sin 2(\psi^{b}_{pol} + \psi) \right] \qquad (14)}$$