Appendix: Conversion Factors and Units

We have now presented 4 different ways to give the Stokes parameters:

1) $I(\vec{q}),\ Q(\vec{q}),\ U(\vec{q})$ are dimensionless numbers that refer to occupation numbers of quantum states.

2) $I_A = T_A,\ Q_A,\ U_A$ are corresponding antenna temperatures (SI unit is Kelvin)

3) $I_B = B_\nu,\ Q_B,\ U_B$ express them in terms of radiation brightness, (unit is $\frac{W}{m^2 \cdot Hz \cdot sr}$ or $\frac{Jy}{sr}$)

4) $I_c = \Theta,\ Q_c,\ U_c$ are relative perturbations of the CMB temperature and are dimensionless numbers.

For present day observations, they may also be given as absolute perturbations of the CMB temperature, i.e., multiplied by the present background temperature $T_0 = 2.725\,K$:

$$I_c^{abs} = \Theta^{abs} \equiv \Theta T_0 \ ; \quad Q_c^{abs} \equiv Q_c T_0 \ ; \quad U_c^{abs} \equiv U_c T_0$$

Then the SI unit is Kelvin.

- The conversion factors between these depend on the frequency (photon energy); and for case 4 also on the CMB background temperature $T(\eta)$. Note that in case 4) the Stokes parameters refer to just the perturbation part of the radiation. Between cases 1-3 the conversion factor is the same whether we are talking about the full quantity or the perturbation part.

- Consider first the cases 1-3: From Eqs (8.9) and (9.11) we have ($C = I, Q, U$, or V)

$$\boxed{\begin{aligned} C_A &= \frac{h\nu}{2k_B} C(\vec{q}) &&= \frac{1}{2k_B}\left(\frac{c^2}{\nu^2}\right) C_B \\ C_B &= \frac{h}{c^2}\nu^3 C(\vec{q}) &&= 2k_B\left(\frac{\nu^2}{c^2}\right) C_A \\ C(\vec{q}) &= \frac{2k_B}{h\nu} C_A &&= \frac{c^2}{h\nu^3} C_B \end{aligned}}$$

- Using the CODATA 2006 values

$$c = 299\,792\,458 \ \text{m/s}$$

$$h = 6.626\,068\,96 \times 10^{-34} \ \text{Js} = 4.135\,667\,33 \times 10^{-15} \ \text{eVs}$$

$$k_B = 1.380\,6504 \times 10^{-23} \ \text{J/K} = 8.617\,343 \times 10^{-5} \ \text{eV/K}$$

and using $\nu = 100\ \text{GHz} = 10^{11}\ \text{Hz}$ as a reference value,

$$h\nu = 6.626\,068\,960 \times 10^{-23}\ \text{J} = 4.135\,667\,330 \times 10^{-4}\ \text{eV}$$

$$\frac{h\nu}{k_B} = 4.799\,237\,345\ \text{K} \Rightarrow \underline{\frac{h\nu}{2k_B} = 2.399\,618\,672\ \text{K}} \ ; \ \underline{\frac{2k_B}{h\nu} = 0.416\,732\,8799\ \text{K}^{-1}}$$

$$\lambda = \frac{c}{\nu} = 2.997\,924\,580 \times 10^{-3}\ \text{m} = 2.997\,924\,580\ \text{mm}$$

$$\left(\frac{c^2}{\nu^2}\right) = 8.987\,551\,787 \times 10^{-6}\ \text{m}^2 \qquad 1\ \text{Jy} \equiv 10^{-26} \frac{\text{W}}{\text{m}^2\,\text{Hz}}$$

$$\underline{\frac{1}{2k_B}\left(\frac{c^2}{\nu^2}\right)} = 3.254\,825\,330 \times 10^{17}\ \frac{\text{K}}{\text{J}}\text{m}^2 = \underline{3.254\,825\,330 \times 10^{-9}\ \frac{\text{sr}\cdot\text{K}}{\text{Jy}}}$$

$$\left(\frac{\nu^2}{c^2}\right) = 1.112\,650\,056 \times 10^{5}\ \text{m}^{-2} \qquad \left(\text{m}^{-2} = \frac{1}{\text{m}^2.\text{Hz}\cdot\text{s}\cdot\text{sr}} \ ; \ \frac{\text{J}}{\text{m}^2} = \frac{\text{W}}{\text{m}^2.\text{Hz}.\text{sr}}\right)$$

$$\underline{h\nu.\left(\frac{\nu^2}{c^2}\right)} = 7.372\,496\,000 \times 10^{-18}\ \frac{\text{W}}{\text{m}^2.\text{Hz}\cdot\text{sr}} = \underline{7.372\,496\,000 \times 10^{8}\ \frac{\text{Jy}}{\text{sr}}}$$

$$\underline{2k_B\left(\frac{\nu^2}{c^2}\right)} = 3.072\,361\,490 \times 10^{-18}\ \frac{\text{J}}{\text{m}^2\text{K}} = \underline{3.072\,361\,490 \times 10^{8}\ \frac{\text{Jy}}{\text{sr}\cdot\text{K}}}$$

$$\underline{\frac{c^2}{h\nu^3}} = 1.356\,392\,733 \times 10^{17}\ \frac{\text{m}^2}{\text{J}} = \underline{1.356\,392\,733 \times 10^{-9}\ \frac{\text{sr}}{\text{Jy}}}$$

we get the conversion factors

$$\boxed{\begin{aligned}
C_A &= C(\vec{q})\cdot\left(\frac{\nu}{100\,\text{GHz}}\right)\cdot 2.399\,618\,672\ \text{K} \\
&= \left[\frac{C_B}{\text{Jy/sr}}\right]\cdot\left(\frac{\nu}{100\,\text{GHz}}\right)^{-2}\cdot 3.254\,825\,330 \times 10^{-9}\ \text{K} \\
C_B &= C(\vec{q})\cdot\left(\frac{\nu}{100\,\text{GHz}}\right)^{3}\cdot 7.372\,496\,000 \times 10^{8}\ \frac{\text{Jy}}{\text{sr}} \\
&= \left[\frac{C_A}{\text{K}}\right]\cdot\left(\frac{\nu}{100\,\text{GHz}}\right)^{2}\cdot 3.072\,361\,490 \times 10^{8}\ \frac{\text{Jy}}{\text{sr}} \\
C(\vec{q}) &= \left[\frac{C_A}{\text{K}}\right]\cdot\left(\frac{\nu}{100\,\text{GHz}}\right)^{-1}\cdot 0.416\,732\,8799 \\
&= \left[\frac{C_B}{\text{Jy/sr}}\right]\cdot\left(\frac{\nu}{100\,\text{GHz}}\right)^{-3}\cdot 1.356\,392\,733 \times 10^{-9}
\end{aligned}}$$

CMB and Antenna Temperature

- For CMB perturbations Θ, Q_c, U_c the physical quantity we are referring to is the perturbation of the CMB temperature. When we want to express them in antenna temperature or brightness terms, the relevant quantity is the perturbation in the antenna temperature δC_A or brightness δC_B. The conversion factors between the perturbations δC_A, δC_B, and $\delta C(\vec{q})$ are the same as for the full quantities; but not so when we convert to/from CMB temperature perturbation units. The conversion between antenna and CMB temperature perturbations is usually needed for observations, so we assume here the present CMB background temperature $T_0 = 2.725\,K$, and refer to CMB absolute temperature perturbations, $\delta T = \delta T_{CMB} = T_0\Theta, T_0 Q_c, T_0 U_c$.

- Convert first the background temperature T_0 to antenna temperature $T_A(\nu)$ at different ν:

$$\boxed{T_A(\nu) \overset{\text{(Eq. P9.10)}}{=} \frac{h\nu}{2k_B}\bar{I}(\vec{q}) \overset{\text{(Eq. P10.4)}}{=} \frac{h\nu}{k_B}\frac{1}{e^{h\nu/k_B T_0}-1} = \frac{x}{e^x-1}T_0}$$

where $\boxed{x \equiv \frac{h\nu}{k_B T_0}} = 1.761\,188\,017\times10^{-11}\left(\frac{\nu}{\mathrm{Hz}}\right) = \underline{1.761\,188\,017\times10^{-2}\left(\frac{\nu}{\mathrm{GHz}}\right)}$. For perturbations:

$$\boxed{\delta T_A(\nu) = \underbrace{\frac{\partial}{\partial T}\left[\frac{h\nu}{k_B}\cdot\frac{1}{e^{h\nu/k_B T}-1}\right]_{T=T_0}}_{\frac{h\nu}{k_B}\cdot\frac{e^x}{(e^x-1)^2}\cdot\frac{h\nu}{k_B T_0^2} = \frac{x^2e^x}{(e^x-1)^2}} \cdot \delta T = \left(\frac{x}{e^{x/2}-e^{-x/2}}\right)^2\cdot\delta T}$$

(This is the same calculation as in §P10, except for the conversion factor $\frac{h\nu}{2k_B} = \frac{xT_0}{2}$ from $C(\vec{q})$ to C_A.)

- For the central frequencies of the 9 Planck channels, we get the following table

ν	$x = \frac{h\nu}{k_B T_0}$	$\frac{T_A}{T_0} = \frac{x}{e^x-1}$	$T_A(K)$	$\frac{\delta T_A}{\delta T_{CMB}} = \left(\frac{x}{e^{x/2}-e^{-x/2}}\right)^2$	$\frac{\delta T_{CMB}}{\delta T_A} = \left(\frac{e^{x/2}-e^{-x/2}}{x}\right)^2$
30 GHz	0.528 356 4050	0.758 977 6493	2.068 214 094 K	0.977 057 7743	1.023 480 931
44	0.774 922 7273	0.662 086 9517	1.804 186 944	0.951 425 3580	1.051 054 601
70	1.232 831 612	0.507 143 8406	1.381 966 966	0.882 417 8333	1.133 249 989
100	1.761 188 017	0.365 441 2311	0.995 827 3548	0.777 158 0103	1.286 739 616
143	2.518 498 864	0.220 728 2293	0.601 484 4248	0.604 624 7459	1.653 918 413
217	3.821 777 996	.085 526 39181	0.233 059 4177	0.334 177 6460	2.992 420 385
353	6.216 993 698	.012 429 159 32	0.033 869 459 14	.077 426 489 16	12.915 476 48
545	9.598 474 690	$6.511\,290\,798\times10^{-4}$	$1.774\,326\,743\times10^{-3}$	$6.250\,269\,962\times10^{-3}$	159.993 0893
857	15.093 381 30	$4.205\,469\,157\times10^{-6}$	$1.145\,990\,345\times10^{-5}$	$6.347\,476\,723\times10^{-5}$	15754.29172

- The CMB temperature perturbation scale is physically meaningful only for the CMB. The observed radiation, or its deviation from the isotropic CMB, contains foreground radiation in addition to the CMB perturbation. Thus the measurements at different frequencies are often reported in other units, as brightness deviations or antenna temperature deviations $\delta T_A(\hat{n}_{obs}, \nu)$.

  It is the task of Component Separation, to separate out from those $\delta T_A(\hat{n}_{obs}, \nu)$ (for I, Q, and U) at different ν, the CMB anisotropy/polarization component which is frequency-independent at the CMB scale, $\delta T_{CMB}(\hat{n}_{obs})$ for I, Q, and U, and the foreground components. These foreground components are then delivered to the astronomers in units of Jy/sr; or, for point sources, in Jy.

- The conversion from the CMB perturbation scale to Jy/sr, can be done as $\delta T_{CMB} \to \delta T_A \to C_B$ using the results in the preceding tables. In general

$$\delta C_B = 2k_B\left(\frac{\nu^2}{c^2}\right)\delta T_A = 2k_B\left(\frac{\nu^2}{c^2}\right)\left(\frac{x}{e^{x/2}-e^{-x/2}}\right)^2 \delta T_{CMB}$$

$$= \left(\frac{\nu}{100\,\mathrm{GHz}}\right)^2 \cdot \left(\frac{x}{e^{x/2}-e^{-x/2}}\right)^2 \cdot \left(\frac{\delta T_{CMB}}{\mathrm{K}}\right) \cdot 3.072\,361\,490 \times 10^8 \,\frac{\mathrm{Jy}}{\mathrm{sr}}$$

and, for example, for 100 GHz

$$\delta C_B = \left(\frac{\delta T_{CMB}}{\mathrm{K}}\right) \times 1.773\,024\,161 \times 10^9 \,\frac{\mathrm{Jy}}{\mathrm{sr}} = \left(\frac{\delta T_{CMB}}{\mu\mathrm{K}}\right) \times 1773.024\,161 \,\frac{\mathrm{Jy}}{\mathrm{sr}}$$