C7. Multipole Expansion

- We now expand the directional dependence of the temperature anisotropy and polarization fields in spherical harmonics:

$$\Theta(\eta,\hat{n}) = \sum_{lm} a^T_{lm}(\eta)\, Y_l^m(\hat{n}) = \sum (-i)^l \sqrt{\frac{4\pi}{2l+1}}\, \Theta_l^m(\eta)\, Y_l^m(\hat{n})$$

$$(Q+iU)(\eta,\hat{n}) = \sum_{lm} a_{2,lm}(\eta)\, {}_2Y_l^m(\hat{n}) = \sum (-i)^l \sqrt{\frac{4\pi}{2l+1}} \left[E_l^m(\eta) + iB_l^m(\eta)\right] {}_2Y_l^m(\hat{n}) \qquad (1)$$

$$(Q-iU)(\eta,\hat{n}) = \sum_{lm} a_{-2,lm}(\eta)\, {}_{-2}Y_l^m(\hat{n}) = \sum (-i)^l \sqrt{\frac{4\pi}{2l+1}} \left[E_l^m(\eta) - iB_l^m(\eta)\right] {}_{-2}Y_l^m(\hat{n})$$

where we have defined

$$\Theta_l^m \equiv i^l \sqrt{\frac{2l+1}{4\pi}}\, a^T_{lm} = i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega\, Y_l^{m*}(\hat{n})\,\Theta(\hat{n})$$

$$E_l^m \equiv -i^l \sqrt{\frac{2l+1}{4\pi}}\, a^E_{lm} = i^l \sqrt{\frac{2l+1}{4\pi}} \cdot \frac{1}{2}\left(a_{2,lm} + a_{-2,lm}\right) = i^l \sqrt{\frac{2l+1}{2\pi}} \cdot a^G_{lm} \qquad (2)$$

$$B_l^m \equiv -i^l \sqrt{\frac{2l+1}{4\pi}}\, a^B_{lm} = -i^l \sqrt{\frac{2l+1}{4\pi}}\, \frac{i}{2}\left(a_{2,lm} - a_{-2,lm}\right) = i^l \sqrt{\frac{2l+1}{2\pi}} \cdot a^C_{lm}$$

so that

$$E_l^m + iB_l^m = -i^l \sqrt{\frac{2l+1}{4\pi}} \left(a^E_{lm} + i a^B_{lm}\right) = i^l \sqrt{\frac{2l+1}{4\pi}}\, a_{2,lm} = i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega\, {}_2Y_l^{m*}(\hat{n})\,(Q+iU)(\hat{n}) \qquad (3)$$

$$E_l^m - iB_l^m = -i^l \sqrt{\frac{2l+1}{4\pi}} \left(a^E_{lm} - i a^B_{lm}\right) = i^l \sqrt{\frac{2l+1}{4\pi}}\, a_{-2,lm} = i^l \sqrt{\frac{2l+1}{4\pi}} \int d\Omega\, {}_{-2}Y_l^{m*}(\hat{n})\,(Q-iU)(\hat{n})$$

- We can now pick the multipoles of the Boltzmann eq. (3.14) collision terms:

The Loss Term

- The loss term $\frac{\partial}{\partial y}T(y,\hat{n}) = -an_e\sigma_T \cdot T(y,\hat{n})$ (4)

has the same angular dependence on both sides, so for the multipoles it becomes simply

$$(\,' \equiv \tfrac{d}{dy})$$

$$\Theta_L^{m\,\prime} = -an_e\sigma_T\,\Theta_L^m$$

$$E_L^{m\,\prime} = -an_e\sigma_T\,E_L^m \qquad (5)$$

$$B_L^{m\,\prime} = -an_e\sigma_T\,B_L^m$$

Doppler Effect

- The Doppler shift $\frac{\partial\Theta}{\partial y} = an_e\sigma_T\,\hat{n}\cdot\vec{v}_b = an_e\sigma_T\,n_i v_b^i$ (6)

affects only the temperature anisotropy. We divide the baryon velocity perturbation into its scalar $(\vec{v}\parallel\hat{k}=\hat{z})$ and vector $(\vec{v}\perp\hat{k}=\hat{z})$ parts as

$$v_b^{(0)} = v_b^3$$

(This is $= -iv_b(\vec{k})$, where v_b is the scalar velocity perturbation defined in our discussion of Cosmological Perturbation Theory.)

$$v_b^{(+1)} = \frac{-1}{\sqrt{2}}(v_b^1 - iv_b^2) \qquad\Rightarrow\qquad v_b^1 = \frac{1}{\sqrt{2}}\left(v_b^{(-1)} - v_b^{(+1)}\right) \qquad (7)$$

$$v_b^{(-1)} = \frac{1}{\sqrt{2}}(v_b^1 + iv_b^2) \qquad\qquad v_b^2 = \frac{-i}{\sqrt{2}}\left(v_b^{(-1)} + v_b^{(+1)}\right)$$

$$\Rightarrow\quad n_i v_b^i = n_3 v_b^{(0)} + n_1\cdot\frac{1}{\sqrt{2}}\left[v_b^{(-1)} - v_b^{(+1)}\right] + n_2\frac{-i}{\sqrt{2}}\left[v_b^{(-1)} + v_b^{(+1)}\right]$$

$$= n_3 v_b^{(0)} + \frac{1}{\sqrt{2}}(n_1 - in_2)v_b^{(-1)} + \frac{1}{\sqrt{2}}(-n_1 - in_2)v_b^{(+1)}$$

$$= \sqrt{\frac{4\pi}{3}}\,Y_1^0(\hat{n})\,v_b^{(0)} + \sqrt{\frac{4\pi}{3}}\,Y_1^{-1}(\hat{n})\,v_b^{(-1)} + \sqrt{\frac{4\pi}{3}}\,Y_1^1(\hat{n})\,v_b^{(+1)} \qquad (8)$$

(see Eq. (F6.6) for the $Y_L^m(\hat{n})$ in terms of n_i), which reveals the multipoles of Eq. (6) as

$$\Theta_1^{0\,\prime} = i\sqrt{\frac{3}{4\pi}}\,a_{10}^{T\,\prime} = an_e\sigma_T\cdot iv_b^{(0)}$$

$$\Theta_1^{1\,\prime} = i\sqrt{\frac{3}{4\pi}}\,a_{11}^{T\,\prime} = an_e\sigma_T\cdot iv_b^{(+1)}$$

$$\Theta_1^{-1\,\prime} = i\sqrt{\frac{3}{4\pi}}\,a_{1,-1}^{T\,\prime} = an_e\sigma_T\cdot iv_b^{(-1)}$$

or, in one equation,

$$\underline{\Theta_1^{m\,\prime} = an_e\sigma_T\cdot iv_b^{(m)}} \qquad (9)$$

The Gain Term

- The multipoles of the gain term can be picked from Eq. (6.12):

$$
\begin{aligned}
a_{00}^{T\,\prime} &= a n_e \sigma_T \cdot a_{00}^{T} \\
a_{2m}^{T\,\prime} &= a n_e \sigma_T \cdot \frac{1}{10}\left(a_{2m}^{T} + \sqrt{6}\, a_{2m}^{E}\right) \\
a_{2,2m}^{\prime} &= a n_e \sigma_T \cdot \frac{1}{10}\left(-\sqrt{6}\, a_{2m}^{T} - 6 a_{2m}^{E}\right) \\
a_{-2,2m}^{\prime} &= a n_e \sigma_T \cdot \frac{1}{10}\left(-\sqrt{6}\, a_{2m}^{T} - 6 a_{2m}^{E}\right) = a_{2,2m}^{\prime}
\end{aligned}
\qquad (10)
$$

and from those,

$$
\boxed{
\begin{aligned}
\Theta_0^{0\,\prime} &= a n_e \sigma_T \cdot \Theta_0^0 \\
\Theta_2^{m\,\prime} &= a n_e \sigma_T \cdot \frac{1}{10}\left(\Theta_2^m - \sqrt{6}\, E_2^m\right) \\
E_2^{m\,\prime} &= i^l \sqrt{\frac{2l+1}{4\pi}} \cdot \frac{1}{2}\left(a_{2,2m}^{\prime} + a_{-2,2m}^{\prime}\right) \\
&= - a n_e \sigma_T \cdot \frac{\sqrt{6}}{10}\, \Theta_2^m + a n_e \sigma_T \cdot \frac{6}{10} E_2^m \\
B_2^{m\,\prime} &= -i^l \sqrt{\frac{2l+1}{4\pi}} \cdot \frac{i}{2}\left(a_{2,2m}^{\prime} - a_{-2,2m}^{\prime}\right) = 0
\end{aligned}
}
\qquad (11)
$$

Summary of the Effect of Thomson Scattering

- The loss term damps all multipoles of temperature anisotropy and polarization at the same rate, $a n_e \sigma_T$.
- Doppler effect generates $l=1$ temperature anisotropy from the baryon velocity perturbation, scalar generating scalar $(m=0)$, and vector generating vector $(m=\pm 1)$.
- The monopole of the gain term exactly cancels the monopole of the loss term
  $\Rightarrow$ Thomson scattering has no effect on the monopole of the temperature perturbation
- The quadrupole of the gain term cancels 10% of the damping of the temperature quadrupole, and 60% of the damping of the E-mode polarization quadrupole.
- The temperature quadrupole is a source of E-mode polarization (quadrupole).
- The E-mode quadrupole is a source of temperature anisotropy (quadrupole).
- Thomson scattering does not mix different l or m modes (consider $\vec{v}_b$ as $L=1$).

- There is no gain term for B-mode polarization. That is, no matter what the nature of the incoming radiation — even if the incoming radiation has B-mode polarization — the outgoing radiation has no B-mode polarization. Thomson scattering does not produce any B-mode. It just damps any pre-existing B-mode polarization.

$\therefore$ $\boxed{\text{Thomson scattering produces E-mode polarization from temperature anisotropy (from its quadrupole).}}$

What's left

- The other terms, besides the collision terms (from Thomson scattering; Doppler, loss, and gain) in the photon Boltzmann equation (3.14); are 1) gravitational redshift 2) free streaming.
- We did those for the temperature anisotropy Θ in Chapter F.
- As already noted in §C3, gravitational redshift has no effect on polarization.
- Thus only the free streaming of polarization remains, to be done, to complete the photon Boltzmann equations for the multipoles. This is left as an exercise (hint: you can use the CG series result (5.6)).