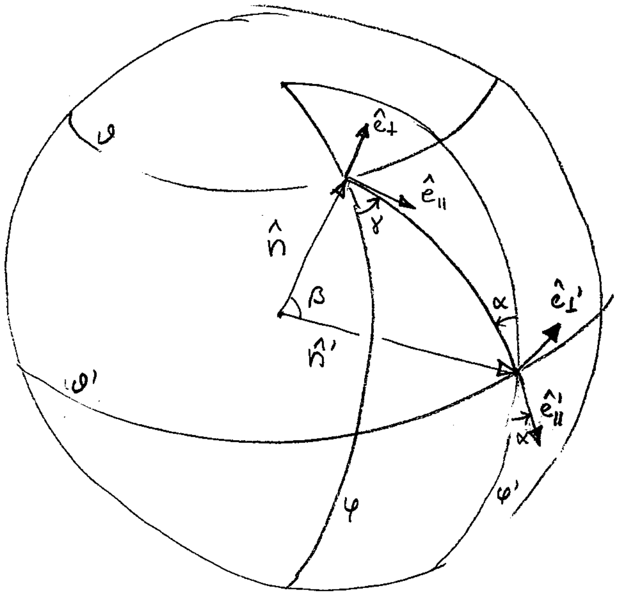
# C6. Rotation of the Collision Term

- In the collision term $R(-\gamma)\,S(\beta)\,R(\alpha)$, the angles $\alpha$ and $-\gamma$ rotate between the $(\hat\vartheta', \hat\varphi')$, $(\hat\vartheta, \hat\varphi)$ bases and the $(\hat e_{\parallel}', \hat e_{\perp}')$, $(\hat e_{\parallel}, \hat e_{+})$ bases for polarization at $\hat n'$ and $\hat n$.




We need a relation between these rotation angles and the direction vectors

$\hat n = (\vartheta, \varphi)$

$\hat n' = (\vartheta', \varphi')$

- From (1.17) and (3.7), the rotated scattering matrix is $R(-\gamma)\,S(\beta)\,R(\alpha) =$

$$= \frac{3}{4}\begin{bmatrix} 1 & & \\ & e^{i2\gamma} & \\ & & e^{-i2\gamma}\end{bmatrix}\begin{bmatrix} 1+\cos^2\beta & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\sin^2\beta & \frac{1}{2}(1+\cos\beta)^2 & \frac{1}{2}(1-\cos\beta)^2 \\ -\sin^2\beta & \frac{1}{2}(1-\cos\beta)^2 & \frac{1}{2}(1+\cos\beta)^2 \end{bmatrix}\begin{bmatrix} 1 & & \\ & e^{-i2\alpha} & \\ & & e^{+i2\alpha}\end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4}(1+\cos^2\beta) & \left(-\frac{3}{8}\sin^2\beta\right)e^{-i2\alpha} & \left(-\frac{3}{8}\sin^2\beta\right)e^{i2\alpha} \\ e^{i2\gamma}\left(-\frac{3}{4}\sin^2\beta\right) & e^{i2\gamma}\cdot\frac{3}{8}(1+\cos\beta)^2\,e^{-i2\alpha} & e^{i2\gamma}\cdot\frac{3}{8}(1-\cos\beta)^2\,e^{i2\alpha} \\ e^{-i2\gamma}\left(-\frac{3}{4}\sin^2\beta\right) & e^{-i2\gamma}\cdot\frac{3}{8}(1-\cos\beta)^2\,e^{-i2\alpha} & e^{-i2\gamma}\cdot\frac{3}{8}(1+\cos\beta)^2\,e^{i2\alpha} \end{bmatrix}$$

We recognize the matrix elements as $L=2$ Wigner D-functions

$$= \begin{bmatrix} 1+\frac{1}{2}D^2_{00}(\alpha,\beta,-\gamma) & -\frac{1}{2}\sqrt{\frac{3}{2}}\,D^2_{20}(\alpha,\beta,-\gamma) & -\frac{1}{2}\sqrt{\frac{3}{2}}\,D^2_{-2,0}(\alpha,\beta,-\gamma) \\ -\sqrt{\frac{3}{2}}\,D^2_{02}(\alpha,\beta,-\gamma) & \frac{3}{2}D^2_{22}(\alpha,\beta,-\gamma) & \frac{3}{2}D^2_{-2,2}(\alpha,\beta,-\gamma) \\ -\sqrt{\frac{3}{2}}\,D^2_{0,-2}(\alpha,\beta,-\gamma & \frac{3}{2}D^2_{2,-2}(\alpha,\beta,-\gamma) & \frac{3}{2}D^2_{-2,-2}(\alpha,\beta,-\gamma) \end{bmatrix} \quad (1)$$

(see Varshalovich et al. Table 4.6 on p. C4.3)

- The problem with Eq. (1) is that it is in terms of the angles $\alpha, \beta, \gamma$ (which depend on both $\hat{n} = (\vartheta, \varphi)$ and $\hat{n}' = (\vartheta', \varphi')$), whereas in Eq. (3.14) we need to integrate it over $\vartheta', \varphi'$. We need to express Eq. (1) in terms of $\vartheta, \varphi, \vartheta', \varphi'$.

- The rotation group SO(3), which we consider as the set of crd. transformations $x, y, z \to x', y', z'$ (or $\vartheta, \varphi \to \vartheta', \varphi'$) between orthonormal right-handed crd. systems, and its parametrization in terms of the Euler angles $\alpha, \beta, \gamma$, were mentioned in §C4. Read now the definition/description of Euler angles in Appendix p. DB-3.

- We denote rotation around an axis $\hat{a}$ by an angle $\alpha$ as $R(\hat{a}, \alpha)$, and the rotation by Euler angles $\alpha, \beta, \gamma$ as $R(\alpha, \beta, \gamma)$. Doing first rotation $R_1$ and then rotation $R_2$, is the multiplication operation of the rotation group and we denote the resulting total rotation as

$$R_{tot} = R_1 R_2 \tag{2}$$

The rotation group is non-Abelian, i.e., the multiplication does not commute, i.e.,

$$R_1 R_2 \neq R_2 R_1 \tag{3}$$

so we had to decide which of these two notations to use to denote doing first $R_1$ and then $R_2$. Both conventions are used, but for the present purpose we chose the convention where the succession goes from left to right; since that corresponds to the multiplication order of the Wigner matrices $D^{(l)}$, Eq. (4.15).

- Thus, from the definition of the Euler angles,

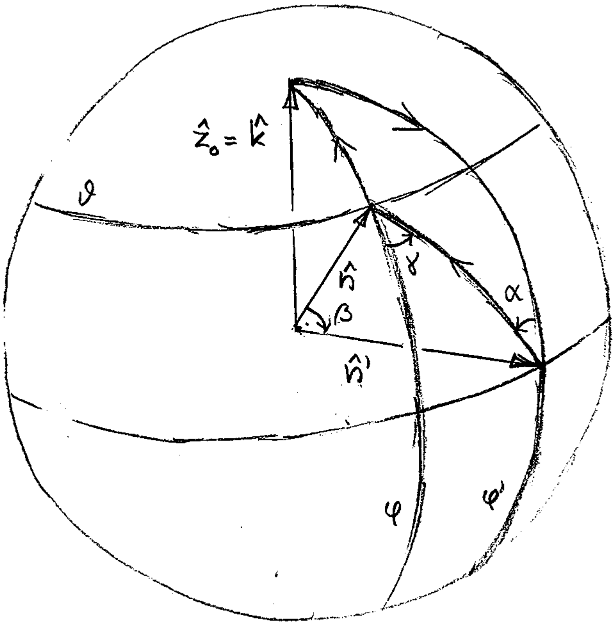
$$R(\alpha, \beta, \gamma) \equiv R(\hat{z}_n, \alpha)\, R(\hat{y}_{n+1}, \beta)\, R(\hat{z}_{n+2}, \gamma) \tag{4}$$

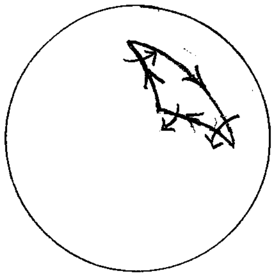
which rotates from the crd. system $x_n, y_n, z_n$ to $x_{n+3}, y_{n+3}, z_{n+3}$.

- The inverse rotation of $R(\alpha, \beta, \gamma)$ is

$$R^{-1}(\alpha, \beta, \gamma) = R(-\gamma, -\beta, -\alpha) \tag{5}$$

as already mentioned on p. C4.2.



- We can relate the angles $\alpha, \beta, \gamma$ to $\vartheta, \varphi, \vartheta', \varphi'$ by considering a sequence of rotations. We start from the original crd. system $x_0, y_0, z_0$, in which the angles $\varphi, \vartheta, \varphi', \vartheta'$ are defined, and in which $\hat{z}_0 = \hat{k}$ (the Fourier mode wave vector). We then rotate so that the $\hat{z}$ axis goes first to $\hat{n}'$, then to $\hat{n}$, and then back to $\hat{k}$. Finally we rotate around this $\hat{z} = \hat{k}$ so that also the $\hat{x}$ and $\hat{y}$ axes return to the original $\hat{x}_0, \hat{y}_0$. As demonstrated during the lecture, this is achieved by

$$\underbrace{R(\hat{z}_0,\varphi')R(\hat{y}_1,\vartheta')}_{R(\varphi',\vartheta',0)}\underbrace{R(\hat{z}_2,\alpha)R(\hat{y}_3,-\beta)R(\hat{z}_4,-\gamma)}_{R(\alpha,-\beta,-\gamma)}\underbrace{R(\hat{y}_5,-\vartheta)R(\hat{z}_6,-\varphi)}_{R(0,-\vartheta,-\varphi)} = \mathbb{1} \qquad (6)$$

The total rotation is an identity operation, since the final crd. system is equal to the original one, $\hat{x}_7 = \hat{x}_0$, $\hat{y}_7 = \hat{y}_0$, $\hat{z}_7 = \hat{z}_0$. Using (5), we multiply Eq. (6)

on the left by $\quad R^{-1}(\varphi',\vartheta',0) = R(0,-\vartheta',-\varphi')$

and on the right by $\quad R^{-1}(0,-\vartheta,-\varphi) = R(\varphi,\vartheta,0)$

to get

$$\underline{R(\alpha,-\beta,-\gamma) = R(0,-\vartheta',-\varphi')R(\varphi,\vartheta,0)} \qquad (7)$$

- Since the Wigner matrices $D^{(L)}$ are a representation of the rotation group, Eq. (7) implies

$$\underbrace{D^{L}_{mm''}(\alpha,-\beta,-\gamma)}_{\text{Eq. (4.11)} \Rightarrow D^{L\,*}_{-m,-m''}(\alpha,\beta,-\gamma)} = \sum_{m'} \underbrace{D^{L}_{mm'}(0,-\vartheta',-\varphi')}_{D^{L\,*}_{m'm}(\varphi',\vartheta',0) \text{ from Eq. (4.8)}} D^{L}_{m'm''}(\varphi,\vartheta,0) \tag{8}$$

Taking the complex conjugate and renaming $-m \to s_1$, $-m'' \to s_2$, $m' \to m$, we obtain

$$\boxed{\underline{D^{L}_{s_1 s_2}(\alpha,\beta,-\gamma)} = \sum_m D^{L}_{m,-s_1}(\varphi',\vartheta',0)\, D^{L\,*}_{m,-s_2}(\varphi,\vartheta,0) = \underline{\frac{4\pi}{2L+1} \sum_m {}_{s_1}Y_L^{m\,*}(\vartheta',\varphi')\, {}_{s_2}Y_L^{m}(\vartheta,\varphi)}} \tag{9}$$

the (generalized) <u>Addition Theorem</u> for spin-s spherical harmonics. Taking $s_1 = s_2 = 0$, this becomes the addition theorem of ordinary $Y_L^m$:

$$\underline{\sum_m Y_L^m(\vartheta',\varphi')^* \, Y_L^m(\vartheta,\varphi)} = \frac{2L+1}{4\pi} D^{L}_{00}(0,\beta,0) = \sqrt{\frac{2L+1}{4\pi}}\, Y_L^0(\beta,0) = \underline{\frac{2L+1}{4\pi} P_L(\cos\beta)} \tag{10}$$

- Using (9), we can now write (1) in the desired form

$$R(-\gamma)\,S(\beta)\,R(\alpha) = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} +$$

$$+ \frac{4\pi}{10} \sum_m \underbrace{\begin{bmatrix} {}_0Y_2^{m*}(\hat n')\,{}_2Y_2^m(\hat n) & -\sqrt{\tfrac{3}{2}}\,{}_2Y_2^{m*}(\hat n')\,{}_0Y_2^m(\hat n) & -\sqrt{\tfrac{3}{2}}\,{}_{-2}Y_2^{m*}(\hat n')\,{}_0Y_2^m(\hat n) \\ -\sqrt{6}\,{}_0Y_2^{m*}(\hat n')\,{}_2Y_2^m(\hat n) & 3\,{}_2Y_2^{m*}(\hat n')\,{}_2Y_2^m(\hat n) & 3\,{}_{-2}Y_2^{m*}(\hat n')\,{}_2Y_2^m(\hat n) \\ -\sqrt{6}\,{}_0Y_2^{m*}(\hat n')\,{}_{-2}Y_2^m(\hat n) & 3\,{}_2Y_2^{m*}(\hat n')\,{}_{-2}Y_2^m(\hat n) & 3\,{}_{-2}Y_2^{m*}(\hat n')\,{}_{-2}Y_2^m(\hat n) \end{bmatrix}}_{\equiv\, P^{(m)} \,=\, \begin{bmatrix} Y_2^m(\hat n) \\ -\sqrt{6}\,{}_2Y_2^m(\hat n) \\ -\sqrt{6}\,{}_{-2}Y_2^m(\hat n) \end{bmatrix} \begin{bmatrix} Y_2^{m*}(\hat n') & -\sqrt{\tfrac{3}{2}}\,{}_2Y_2^{m*}(\hat n') & -\sqrt{\tfrac{3}{2}}\,{}_{-2}Y_2^{m*}(\hat n') \end{bmatrix}} \tag{11}$$

where $\hat n = (\vartheta,\varphi)$ and $\hat n' = (\vartheta',\varphi')$.

- Now we can finally do the integration in the scattering gain term of Eq. (3.14):

$$\frac{\partial}{\partial y}\begin{bmatrix}\Theta(\hat{n})\\ (Q+iU)(\hat{n})\\ (Q-iU)(\hat{n})\end{bmatrix} = a n_e \sigma_T \int \frac{d\Omega'}{4\pi} R(-\gamma)S(\beta)R(\alpha)\begin{bmatrix}\Theta(\hat{n}')\\ (Q+iU)(\hat{n}')\\ (Q-iU)(\hat{n}')\end{bmatrix}$$

$$= a n_e \sigma_T \begin{bmatrix}\int \frac{d\Omega'}{4\pi}\Theta(\hat{n}')\\ 0\\ 0\end{bmatrix} \qquad \int \frac{d\Omega'}{4\pi}\Theta(\hat{n}') = \frac{1}{\sqrt{4\pi}}a^T_{00} = a^T_{00}\cdot Y_0^0(\hat{n})$$

$$+ a n_e \sigma_T \cdot \frac{1}{10}\sum_m \begin{bmatrix} Y_2^m(\hat{n})\\ -\sqrt{6}\,{}_2Y_2^m(\hat{n})\\ -\sqrt{6}\,{}_{-2}Y_2^m(\hat{n})\end{bmatrix}\cdot \underbrace{\int d\Omega'\left(Y_2^{m*}\Theta - \sqrt{\tfrac{3}{2}}\,{}_2Y_2^{m*}\cdot(Q+iU) - \sqrt{\tfrac{3}{2}}\,{}_{-2}Y_2^{m*}\cdot(Q-iU)\right)}_{=\, a^T_{2m} - \sqrt{\frac{3}{2}}a_{2,2m} - \sqrt{\frac{3}{2}}a_{-2,2m}}$$

(from Eq. Y5.6)

$$= a n_e \sigma_T \cdot \begin{bmatrix} a^T_{00}\cdot Y_0^0(\hat{n}) + \frac{1}{10}\sum_m\left(a^T_{2m} - \sqrt{\tfrac{3}{2}}a_{2,2m} - \sqrt{\tfrac{3}{2}}a_{-2,2m}\right)Y_2^m(\hat{n})\\ \frac{1}{10}\sum_m\left(-\sqrt{6}a^T_{2m} + 3a_{2,2m} + 3a_{-2,2m}\right){}_2Y_2^m(\hat{n})\\ \frac{1}{10}\sum_m\left(-\sqrt{6}a^T_{2m} + 3a_{2,2m} + 3a_{-2,2m}\right){}_{-2}Y_2^m(\hat{n})\end{bmatrix}$$

$$= a n_e \sigma_T \cdot \begin{bmatrix} a^T_{00}\cdot Y_0^0(\hat{n}) + \frac{1}{10}\sum_m\left(a^T_{2m} + \sqrt{6}a^E_{2m}\right)Y_2^m(\hat{n})\\ \frac{1}{10}\sum_m\left(-\sqrt{6}a^T_{2m} - 6a^E_{2m}\right){}_2Y_2^m(\hat{n})\\ \frac{1}{10}\sum_m\left(-\sqrt{6}a^T_{2m} - 6a^E_{2m}\right){}_{-2}Y_2^m(\hat{n})\end{bmatrix} \tag{12}$$

where we used $a^E_{\ell m} \equiv -\frac{1}{2}\left(a_{2,\ell m} + a_{-2,\ell m}\right)$ of Eq. (Y5.8).

- The monopole of the gain term, $a^T_{00}Y_0^0(\hat{n})$, exactly cancels the monopole of the loss term, confirming the result from §C2, that Thomson scattering has no effect on isotropic unpolarized radiation. The rest of Eq. (12) shows, that only the quadrupole temperature anisotropy, and the quadrupole of the E-mode polarization, of the incoming radiation contribute to the gain term of Thomson scattering.