C5. Spin-s Harmonics

- The spin-s harmonics ${}_sY_L^m(\vartheta,\varphi)$ are closely related to the Wigner D-functions. The relation is

$$ {}_sY_L^m(\vartheta,\varphi) = \sqrt{\frac{2L+1}{4\pi}}\, D^{L\,*}_{m,-s}(\varphi,\vartheta,0) = \sqrt{\frac{2L+1}{4\pi}}\, e^{im\varphi}\, d^L_{m,-s}(\vartheta) \qquad (1)$$

$$ D^L_{ms}(\varphi,\vartheta,\psi) = \sqrt{\frac{4\pi}{2L+1}}\, {}_{-s}Y_L^{m\,*}(\vartheta,\varphi)\, e^{-is\psi} \qquad (2)$$

- From the Wigner function symmetry relations we get

$$ {}_sY_L^{-m}(\vartheta,\varphi) = (-1)^{m-s}\, {}_{-s}Y_L^m(\vartheta,\varphi)^* \qquad (3)$$

- In Eqs. (1) and (2) we must have $|s| \le L$, since the Wigner functions only have lower indices for such values. Indeed, the L-values for spin-s harmonics begin from $|s|$; ${}_sY_L^m$ for $L \le |s|$ do not exist.

- From (3), we can also write $\quad {}_{-s}Y_L^m(\vartheta,\varphi) = (-1)^{s-m}\, {}_sY_L^{-m}(\vartheta,\varphi)^* \qquad (4)$

- From the Clebsch-Gordan series (4.16) we get the corresponding result for the product of two spin-s harmonics:

$$ \boxed{{}_{s_1}Y_{L_1}^{m_1}(\vartheta,\varphi)\cdot {}_{s_2}Y_{L_2}^{m_2}(\vartheta,\varphi) = \sqrt{\frac{(2L_1+1)(2L_2+1)}{4\pi}}\sum_L \frac{1}{\sqrt{2L+1}}\langle L_1 m_1 L_2 m_2|Lm\rangle\langle L_1,-s_1,L_2,-s_2|L,-s\rangle\, {}_sY_L^m(\vartheta,\varphi)} \qquad (5)$$

where $m = m_1+m_2$, $s = s_1+s_2$, and $L = |L_1-L_2|,\ldots,L_1+L_2$

A special case needed for free streaming, is

$$ \sqrt{\frac{4\pi}{3}}\, {}_sY_L^m(\vartheta,\varphi)\, Y_1^0(\vartheta,\varphi) = \langle Lm10|L-1,m\rangle\langle L,-s,10|L-1,-s\rangle\sqrt{\frac{2L+1}{2L-1}}\, {}_sY_{L-1}^m $$

$$ + \langle Lm10|Lm\rangle\langle L,-s,10|L,-s\rangle\, {}_sY_L^m + \langle Lm10|L+1,m\rangle\langle L,-s,10|L+1,-s\rangle\sqrt{\frac{2L+1}{2L+3}}\, {}_sY_{L+1}^m $$

$$ = \sqrt{\frac{(L^2-m^2)(L^2-s^2)}{L^2(2L+1)(2L-1)}}\, {}_sY_{L-1}^m - \frac{ms}{L(L+1)}\, {}_sY_L^m + \sqrt{\frac{[(L+1)^2-m^2][(L+1)^2-s^2]}{(L+1)^2(2L+1)(2L+3)}}\, {}_sY_{L+1}^m \qquad (6)$$

where we used (from VMK Table 8.2)

$$ \langle Lm10|L-1,m\rangle = -\sqrt{\frac{(L+m)(L-m)}{L(2L+1)}} \qquad \langle Lm10|Lm\rangle = \frac{m}{\sqrt{L(L+1)}} \qquad (7)$$

$$ \langle Lm10|L+1,m\rangle = \sqrt{\frac{(L+m+1)(L-m+1)}{(2L+1)(L+1)}} $$