C4. Wigner D-Functions

- The Euler angles α, β, γ which specify a crd. transformation $x, y, z \to x', y', z'$ or $\vartheta, \varphi \to \vartheta', \varphi'$ were defined in Appendix DB (p. DB-3).

- The Wigner functions $D^L_{mm'}(\alpha, \beta, \gamma)$ that tell how the multipoles a_{Lm} transform,

$$a_{Lm} = \sum_{m'} D^L_{mm'}(\alpha, \beta, \gamma)\, a'_{Lm'} \tag{1}$$

in such a crd. transformation, were introduced in §F6. (Note that this is the inverse transformation)

- We can think of the set of a_{Lm} for a fixed L as a $2L+1$ component "vector", and the $(2L+1)\times(2L+1)$ matrix $D^{(L)}(\alpha, \beta, \gamma)$ for the matrix of rotating these components. These matrices are unitary

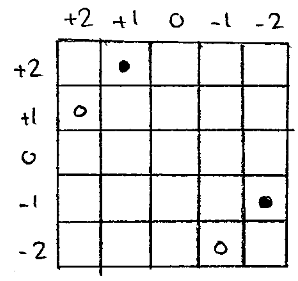
$$D^{(L)}(\alpha, \beta, \gamma)^{\dagger} = D^{(L)}(\alpha, \beta, \gamma)^{-1} \tag{2}$$

- We use the conventions of D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii: Quantum Theory of Angular Momentum (VMK).

- The Wigner D-functions can be given in terms of the real functions $d^L_{mm'}(\beta)$ as

$$D^L_{mm'}(\alpha, \beta, \gamma) = e^{-im\alpha}\, d^L_{mm'}(\beta)\, e^{-im'\gamma} \tag{3}$$

- These d-matrices have a number of symmetries that relate different components:



$$d^L_{mm'}(\beta) = (-1)^{m-m'} d^L_{-m,-m'}(\beta) = (-1)^{m-m'} d^L_{m'm}(\beta) = d^L_{-m',-m}(\beta) \tag{4}$$

- The set of matrices $D^{(L)}(\alpha, \beta, \gamma)$ for a fixed L, forms a representation of the rotation group $SO(3)$. The full set of rotations is covered by the range of Euler angles

$$0 \le \alpha < 2\pi, \qquad 0 \le \beta \le \pi, \qquad 0 \le \gamma < 2\pi \tag{5}$$

We may want to use $D^{(L)}(\alpha, \beta, \gamma)$ with other values of α, β, γ (in practice, negative values). Any value of α or γ is equivalent to the same value mod 2π (*).

*) Wigner functions can be defined also for half-integer values of L, in which case only mod 4π values are equivalent; but we consider only integer values of L.

- Negative values of β (opposite direction of the second rotation) cover the same rotations again, since $(\alpha+\pi, \beta, \gamma-\pi)$ is the same rotation as $(\alpha, -\beta, \gamma)$. (Only the end result of the rotation matters, not the process how we constructed it when we defined the Euler angles).

- The inverse of the (α,β,γ) rotation is the $(-\gamma,-\beta,-\alpha)$ rotation. From (2)

$$D^{(L)}(-\gamma,-\beta,-\alpha) = D^{(L)}(\alpha,\beta,\gamma)^{-1} = D^{(L)}(\alpha,\beta,\gamma)^{\dagger} \qquad (7)$$

$$\Rightarrow \quad D^{L}_{mm'}(-\gamma,-\beta,-\alpha) = D^{L\,*}_{m'm}(\alpha,\beta,\gamma) \qquad (8)$$

$$\Rightarrow \quad d^{L}_{mm'}(-\beta) = d^{L}_{m'm}(\beta) \qquad (9)$$

- From the equivalence of $(\alpha+\pi,\beta,\gamma-\pi)$ and $(\alpha,-\beta,\gamma)$

$$D^{L}_{mm'}(\alpha+\pi,\beta,\gamma-\pi) = D^{L}_{mm'}(\alpha,-\beta,\gamma). \qquad (10)$$

- From the preceding relations follow many more symmetry relations, for example (exercise)

$$\underline{D^{L}_{mm'}(\alpha,\beta,\gamma) = D^{L\,*}_{-m,-m'}(\alpha,-\beta,\gamma)} \qquad (11)$$

- The unitarity condition (2) gives $\quad D^{(L)}(\alpha,\beta,\gamma)^{\dagger} D^{(L)}(\alpha,\beta,\gamma) = \mathbb{1} \qquad (12)$

or,

$$\sum_m D^{L\,*}_{mm'}(\alpha,\beta,\gamma) D^{L}_{mm''}(\alpha,\beta,\gamma) = \delta_{m'm''} \qquad (13)$$

Using this, we can get from the inverse $a'_{Lm} \to a_{Lm}$ transformation (1) the direct transformation $\left(a = Da' \Rightarrow D^{\dagger}a = D^{\dagger}Da' = a' \right)$

$$\sum_m D^{L\,*}_{mm'}(\alpha,\beta,\gamma)\, a_{Lm} = \sum_{mm''} D^{L*}_{mm'}(\alpha,\beta,\gamma) D^{L}_{mm''}(\alpha,\beta,\gamma) a'_{Lm''} = a'_{Lm}$$

$$\Rightarrow \quad a'_{Lm} = \sum a_{Lm} D^{L\,*}_{mm'}(\alpha,\beta,\gamma) \qquad (14)$$

Note that, as matrix multiplication, the $D^{(L)}$ matrix multiplies the a_{Lm} from the right. This means that if we do two coordinate rotation in succession, first $(\alpha_1,\beta_1,\gamma_1)$ and then $(\alpha_2,\beta_2,\gamma_2)$ the Wigner matrix for the total transformation $(\alpha_{tot},\beta_{tot},\gamma_{tot})$ results by multiplication from left to right:

$$\boxed{D^{L}_{mm''}(\alpha_{tot},\beta_{tot},\gamma_{tot}) = \sum_{m'} D^{L}_{mm'}(\alpha_1,\beta_1,\gamma_1) D^{L}_{m'm''}(\alpha_2,\beta_2,\gamma_2)} \qquad (15)$$

This result, which is just the rotation group multiplication operation is called the <u>Addition Theorem</u>. The addition theorem of spherical harmonics is a special case of it.

C4.3

Tables 4.3. — 4.12. Explicit forms of $d^J_{MM'}(\beta)$.

Table 4.3.

$d^{1/2}_{MM'}(\beta)$

$M \backslash M'$	$1/2$	$-1/2$
$1/2$	$\cos\frac{\beta}{2}$	$-\sin\frac{\beta}{2}$
$-1/2$	$\sin\frac{\beta}{2}$	$\cos\frac{\beta}{2}$

Table 4.4.

$d^1_{MM'}(\beta)$

$M \backslash M'$	1	0	-1
1	$\frac{1+\cos\beta}{2}$	$-\frac{\sin\beta}{\sqrt{2}}$	$\frac{1-\cos\beta}{2}$
0	$\frac{\sin\beta}{\sqrt{2}}$	$\cos\beta$	$-\frac{\sin\beta}{\sqrt{2}}$
-1	$\frac{1-\cos\beta}{2}$	$\frac{\sin\beta}{\sqrt{2}}$	$\frac{1+\cos\beta}{2}$

Table 4.5.

$d^{3/2}_{MM'}(\beta)$

$M \backslash M'$	$3/2$	$1/2$	$-1/2$	$-3/2$
$3/2$	$\cos^3\frac{\beta}{2}$	$-\sqrt{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}$	$\sqrt{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}$	$-\sin^3\frac{\beta}{2}$
$1/2$	$\sqrt{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}$	$\cos\frac{\beta}{2}\left(3\cos^2\frac{\beta}{2}-2\right)$	$\sin\frac{\beta}{2}\left(3\sin^2\frac{\beta}{2}-2\right)$	$\sqrt{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}$
$-1/2$	$\sqrt{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}$	$-\sin\frac{\beta}{2}\left(3\sin^2\frac{\beta}{2}-2\right)$	$\cos\frac{\beta}{2}\left(3\cos^2\frac{\beta}{2}-2\right)$	$-\sqrt{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}$
$-3/2$	$\sin^3\frac{\beta}{2}$	$\sqrt{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}$	$\sqrt{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}$	$\cos^3\frac{\beta}{2}$

Table 4.6.

$d^2_{MM'}(\beta)$

$M \backslash M'$	2	1	0	-1	-2
2	$\frac{(1+\cos\beta)^2}{4}$	$-\frac{\sin\beta(1+\cos\beta)}{2}$	$\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\beta$	$-\frac{\sin\beta(1-\cos\beta)}{2}$	$\frac{(1-\cos\beta)^2}{4}$
1	$\frac{\sin\beta(1+\cos\beta)}{2}$	$\frac{2\cos^2\beta+\cos\beta-1}{2}$	$-\sqrt{\frac{3}{2}}\sin\beta\cos\beta$	$-\frac{2\cos^2\beta-\cos\beta-1}{2}$	$-\frac{\sin\beta(1-\cos\beta)}{2}$
0	$\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\beta$	$\sqrt{\frac{3}{2}}\sin\beta\cos\beta$	$\frac{3\cos^2\beta-1}{2}$	$-\sqrt{\frac{3}{2}}\sin\beta\cos\beta$	$\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\beta$
-1	$\frac{\sin\beta(1-\cos\beta)}{2}$	$-\frac{2\cos^2\beta-\cos\beta-1}{2}$	$\sqrt{\frac{3}{2}}\sin\beta\cos\beta$	$\frac{2\cos^2\beta+\cos\beta-1}{2}$	$-\frac{\sin\beta(1+\cos\beta)}{2}$
-2	$\frac{(1-\cos\beta)^2}{4}$	$\frac{\sin\beta(1-\cos\beta)}{2}$	$\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\beta$	$\frac{\sin\beta(1+\cos\beta)}{2}$	$\frac{(1+\cos\beta)^2}{4}$

The Clebsch-Gordan Series (product of two D-functions)

$$\boxed{D^{L_1}_{m_1 m_1'}(\alpha,\beta,\gamma)\, D^{L_2}_{m_2 m_2'}(\alpha,\beta,\gamma) = \sum_{L=|L_1-L_2|}^{L_1+L_2} \langle L_1 m_1 L_2 m_2 | L m \rangle\, D^{L}_{m m'}(\alpha,\beta,\gamma)\, \langle L_1 m_1' L_2 m_2' | L m' \rangle} \quad (16)$$

where $m = m_1 + m_2$ and $m' = m_1' + m_2'$

Note that the Clebsch-Gordan coefficients $\langle L_1 m_1 L_2 m_2 | L m \rangle$ are all real.