# C3. The Boltzmann Equation

- From §C2 we got that for radiation, which is unpolarized, and isotropic in the rest frame of the electron fluid, Thomson scattering has no effect, since the loss and gain terms are equal. This is the case we have in the background universe (where the electron fluid is at rest in the comoving cd's of the Friedmann model). Thus Thomson scattering appears in the photon Boltzmann equation only at the perturbation level.

- In the perturbed universe, we have two effects:

  1) The electron fluid is not at rest, but we have a velocity perturbation

     $\vec{v}_e = \vec{v}_b$ (electrons are tightly coupled to protons and the other nuclei, so we have just one common baryon velocity perturbation) (1)

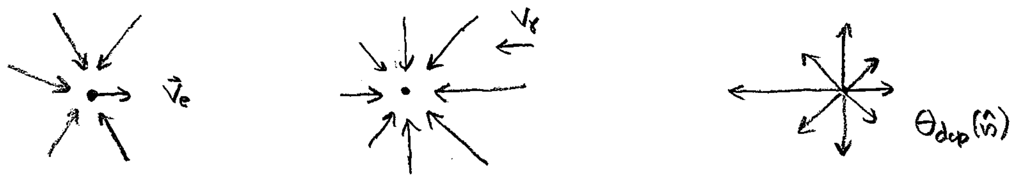
  2) The radiation field is anisotropic and polarized (by the Thomson scattering), so

     $\Theta, Q, U \neq 0$

Since these are perturbations, we can consider their effect separately (in 1st order perturbation theory; the interference between these two effects is 2nd order).

## (1) Doppler Effect

- Consider isotropic unpolarized radiation, with electron fluid moving at nonrelativistic (= 1st order perturbation) velocity $\vec{v}_e$. Transforming to the rest frame of the electron fluid, we now have the photon fluid moving at velocity $\vec{v}_\gamma = -\vec{v}_e$. This is equivalent to a dipole anisotropy*) $\Theta_{dip}(\hat{n}) = \hat{n}\cdot\vec{v}_\gamma = -\hat{n}\cdot\vec{v}_e$ (2)




From §C2 (the exercise), we have that for unpolarized incoming radiation, the loss term damps all multipoles at the same rate, $n_e\sigma_T$, and the gain term is isotropic, if the incoming radiation anisotropy has zero quadrupole. Here we have just a dipole, so the dipole gets damped

$$\frac{d\Theta_{dip}(\hat{n})}{dt} = -n_e\sigma_T\Theta_{dip}(\hat{n}) = n_e\sigma_T(\hat{n}\cdot\vec{v}_e) \qquad (3)$$

Also, only a quadrupole anisotropy produces polarization, so this effect produces no polarization.

*) The non-relativistic Doppler effect. Calculating to higher order, we have also a quadrupole at the second order etc. (the relativistic Doppler effect).

- Transforming back to the original frame, where the electron fluid was moving, this effect on the photon brightness function $\Theta(\hat{n})$ is unchanged to $1^{st}$ order (since it is a $1^{st}$ order effect, and the transformation is also $1^{st}$ order), so we get that in the original frame

$$\frac{\partial \Theta(\hat{n})}{\partial t} = n_e \sigma_T (\hat{n} \cdot \vec{v}_b) \qquad (4)$$

$\therefore$ The effect of the baryon fluid velocity on isotropic radiation is to try to create a dipole in the direction of the baryon flow (i.e., to try to isotropize it in the baryon rest frame).

(2) Effect of Anisotropy and Polarization

- Since the effect of $\vec{v}_e$ (the Doppler effect) was treated separately; we can now ignore it and use the result of §C1, except that now we know that only the perturbation $\Theta$ appears in place of the full Stokes I in Eqs (1.5) and (1.17)
- Since there is no energy ($q$) dependence in the collision term, we can write Eqs (1.5) and (1.17) as equations for just the direction-dependent temperature perturbations $\Theta(\hat{n})$, $Q(\hat{n})$, $U(\hat{n})$) (The $Q$ and $U$ are now the $Q_c$ and $U_c$ of §P10).

The Full Collision Term

- We get now the full collision term

$$\frac{\partial \bar{T}(\hat{n})}{\partial t} = \underbrace{n_e \sigma_T \hat{n} \cdot \vec{v}_b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Doppler}} \underbrace{- n_e \sigma_T \bar{T}(\hat{n})}_{\text{loss}} + \underbrace{n_e \sigma_T \int \frac{d\Omega'}{4\pi} R(-\gamma) S(\beta) R(\alpha) \bar{T}(\hat{n}')}_{\text{gain}} \qquad (5)$$

where $\bar{T} \equiv \begin{bmatrix} \Theta \\ Q+iU \\ Q-iU \end{bmatrix}$ (you can think of the $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as the background version of $\bar{T}$) (6)

is a "vector" representing the anisotropy and polarisation perturbations, and

$$R(\alpha) = \begin{bmatrix} 1 & & \\ & e^{-i2\alpha} & \\ & & e^{+i2\alpha} \end{bmatrix} \qquad R(-\gamma) = \begin{bmatrix} 1 & & \\ & e^{i2\gamma} & \\ & & e^{-i2\gamma} \end{bmatrix} \qquad (7)$$

rotate the Stokes parameters into and out of the scattering plane basis $\hat{e}_{\parallel}, \hat{e}_{\perp}$.

The rotation angles $\alpha$ and $\gamma$, and the scattering angle $\beta$ depend on both $\hat{n}$ and $\hat{n}'$ which define the scattering plane.

- We are now in the position to write the full photon Boltzmann equation (the brightness equation). The earlier version, without the collision term and without polarization was (§F4, Eq. 8)

$$\frac{\partial \Theta}{\partial \eta} = \underbrace{-n^i \frac{\partial \Theta}{\partial x^i}}_{\text{free streaming}} \underbrace{- \tfrac{1}{2} h'_{ij} n^i n^j - h'_{0i} n^i + \tfrac{1}{2} h_{00,i} n^i}_{\text{gravitational redshift}} \qquad (8)$$

- Since both polarization states of the photon redshift the same, the gravitational redshift has no effect on polarization (Stokes Q, U, V).

- Free streaming also affects both polarization states the same. The difference from redshift is that redshift is a source term, whereas free streaming transports the existing anisotropy/polarization; so we get free streaming of polarization

$$\frac{\partial Q}{\partial \eta} = -n^i \frac{\partial Q}{\partial x^i} \qquad \frac{\partial U}{\partial \eta} = -n^i \frac{\partial U}{\partial x^i} \qquad (9)$$

- To include the collision term (5) in this same framework, we include the location dependence of $\bar{T}$, $\bar{T} = \bar{T}(\eta, x^i; \hat{n})$, and switch to conformal time $dt = a\,d\eta$.

  Altogether we have now

$$\boxed{\begin{aligned} \frac{\partial \bar{T}}{\partial \eta} &= -n^i \frac{\partial \bar{T}}{\partial x^i} + \left(-\tfrac{1}{2} h'_{ij} n^i n^j - h'_{0i} n^i + \tfrac{1}{2} h_{00,i} n^i\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &+ a n_e \sigma_T \, \hat{n} \cdot \vec{v}_b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - a n_e \sigma_T \bar{T} + a n_e \sigma_T \int \frac{d\Omega'}{4\pi} R(-\gamma) S(\beta) R(\alpha) \bar{T}(\hat{n}') \end{aligned}} \qquad (10)$$

- We then follow the path of §F5 and §F7, to write this first in Fourier space and then for multipoles.

## Fourier Transform

- Apply this now to a single Fourier mode $\vec{k}$ of the perturbations,

$$\Theta(\eta,\vec{x},\hat{n}) = \Theta(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}}, \qquad \vec{v}_b(\eta,\vec{x},\hat{n}) = \vec{v}_b(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}} \tag{11}$$

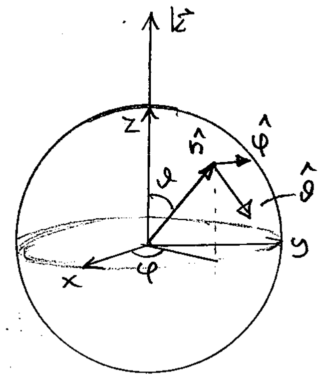
$$Q(\eta,\vec{x},\hat{n}) = Q(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}}, \qquad U(\eta,\vec{x},\hat{n}) = U(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}}$$

and choose the z axis in the $\vec{k}$ direction, so we have $e^{i\vec{k}\cdot\vec{x}} = e^{ikz}$,

$$\bar{T}(\eta,\vec{x},\hat{n}) = \bar{T}(\eta,\hat{n})\, e^{i\vec{k}\cdot\vec{x}} = \bar{T}(\eta,\hat{n})\, e^{ikz} \tag{12}$$

and

$$-n^i \frac{\partial}{\partial x^i} = -i\hat{n}\cdot\vec{k} = -in_3 k \tag{13}$$




- The Stokes parameters for radiation propagating in the direction $\hat{n}$, are defined with respect to the local $\hat{\vartheta}, \hat{\varphi}$ basis.

- The photon Boltzmann eq. is now

free streaming　　　　　　　　grav. redshift

$$\boxed{\begin{aligned} \frac{\partial \bar{T}(\eta,\hat{n})}{\partial \eta} &= -ikn_3\bar{T}(\hat{n}) + \left(-\tfrac{1}{2}h'_{ij}n^i n^j - h'_{0i}n^i + \tfrac{1}{2}ikn_3 h_{00}\right)\begin{bmatrix}1\\0\\0\end{bmatrix} \\ &\quad + an_e\sigma_T\,\hat{n}\cdot\vec{v}_b(\hat{n})\begin{bmatrix}1\\0\\0\end{bmatrix} - an_e\sigma_T\bar{T}(\hat{n}) + an_e\sigma_T\int\frac{d\Omega'}{4\pi}R(-\gamma)S(\beta)R(\alpha)\bar{T}(\hat{n}') \end{aligned}} \tag{14}$$

Doppler　　　　　　　　loss　　　　　　　　gain