C. COLLISIONAL BOLTZMANN EQUATION

C1. Thomson Scattering

- Compton scattering refers to the scattering of photons on electrons

$$\gamma(\vec{q}\,') + e^-(\vec{p}\,') \quad \rightarrow \quad \gamma(\vec{q}) + e^-(\vec{p}) \,. \tag{1}$$

Here $\vec{q}\,', \vec{p}\,'$ are the incoming and $\vec{q}, \vec{p}$ the outgoing momenta. They of course depend on the frame used.

- We shall now use the rest frame of the (incoming) electron, so $\vec{p}\,' = 0$.

For $q' \ll m_e$ we can use the approximation, where the photon energy loss

$q' - q = E_e' - m_e = \sqrt{p'^2 - m_e^2}$ (equal to electron recoil energy) is ignored, so

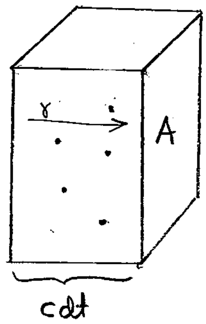
$$\underline{q' = q} \qquad \text{(in the electron rest frame!)}. \tag{2}$$

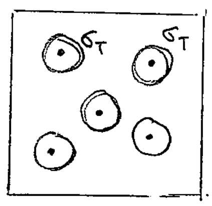
Compton scattering in this approximation is called Thomson scattering.

- The cross section of Thomson scattering is (CODATA 2006)

$$\underline{\sigma_T = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} = 6.652\,458\,558 \times 10^{-29}\ \mathrm{m}^2} \tag{3}$$

This means that for a number density n_e of electrons, the probability for a photon to scatter in time dt, is $\quad n_e c\,\sigma_T\,dt = \underline{n_e \sigma_T\, dt}$ $\quad$ ($c = 1$ = photon velocity) $\quad$ (4)





number of electrons in box $V = A c\,dt$ is

$$n_e V = n_e A c\,dt$$

their cross sections cover fraction of the area A

$$\frac{n_e A c\,dt \cdot \sigma_T}{A} = n_e c\,\sigma_T\,dt$$

Each collision (scattering) leads to a loss of one photon w momentum $\vec{q}\,'$ and a gain of one photon w momentum $\vec{q}$ in the photon distribution function (or density matrix). The loss term for the Boltzmann eq is easy to write, since the loss rate is proportional to f (or g):

$$\frac{df(t, x^i, \vec{q}\,')}{dt} = -n_e(t, x^i)\,\sigma_T \cdot f(t, x^i, \vec{q}\,')$$

Since this holds for all values of $\vec{q}\,'$, we can write it w/o the prime in $\vec{q}$.

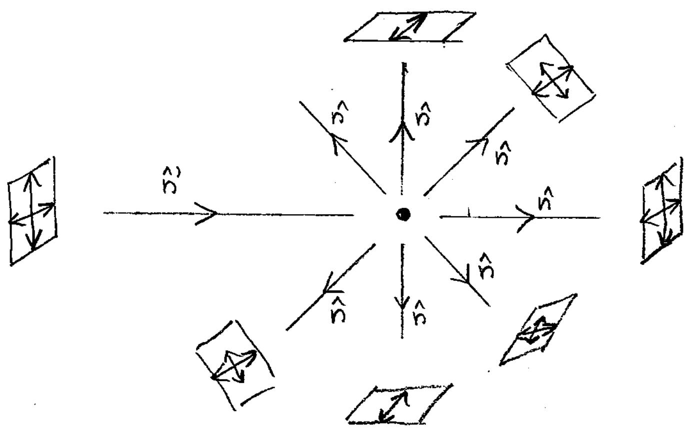
- Thus the loss terms for the Boltzman eqs for the Stokes parameters are

$$\boxed{\begin{aligned} \frac{dI(\vec{q})}{dt} &= -n_e\sigma_T I(\vec{q}) & \frac{dU(\vec{q})}{dt} &= -n_e\sigma_T U(\vec{q}) \\ \frac{dQ(\vec{q})}{dt} &= -n_e\sigma_T Q(\vec{q}) & \frac{dV(\vec{q})}{dt} &= -n_e\sigma_T V(\vec{q}) \end{aligned}} \qquad (5)$$

- The gain term is more complicated, since it is proportional to the number of incoming photons with different $\vec{q}' = q\hat{n}'$ and the probability that such a photon scatters with the outgoing momentum (direction) $\vec{q} = q\hat{n}$. Moreover, we need to consider the relation of the outgoing polarization to the incoming polarization. The differential cross section of Thomson scattering is (Jackson Eq. 14.102)

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi}|\hat{\varepsilon}'^{*}\cdot\hat{\varepsilon}|^2} \qquad (6)$$

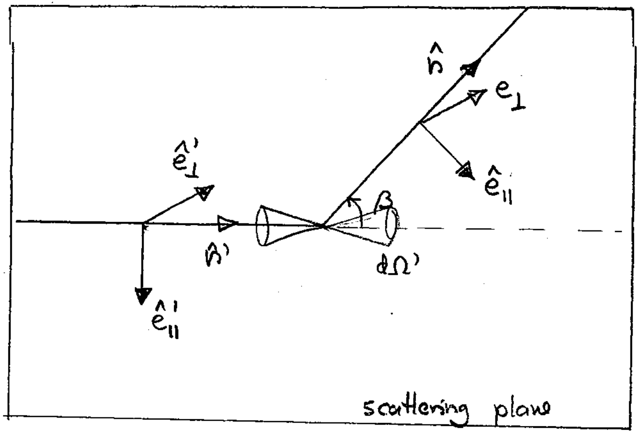
where $\hat{\varepsilon}'$ and $\hat{\varepsilon}$ are the complex unit polarization vectors *) $(\hat{\varepsilon}'\cdot\hat{n}' = 0,\ \hat{\varepsilon}\cdot\hat{n} = 0)$ of the incoming and outgoing photons. In the classical view, they correspond to the oscillation directions of the corresponding electric fields.



We get the maximum probability for scattering where the outgoing polarization $\hat{\varepsilon}$ is parallel to the incoming polarization $\hat{\varepsilon}'$. This is only possible if $\hat{\varepsilon}'$ is orthogonal to the scattering plane, $\hat{\varepsilon}' \perp \hat{n}'$ and $\hat{\varepsilon}' \perp \hat{n}$. There will be no scattering in the direction $\hat{n} = \hat{\varepsilon}'$, since then must have $\hat{\varepsilon}\cdot\hat{\varepsilon}' = \hat{\varepsilon}\cdot\hat{n} = 0$.

*) That is, $\vec{\varepsilon} = (\varepsilon_x, \varepsilon_y) = (a_x e^{i\alpha_x}, a_y e^{i\alpha_y})$ of Eq. (1) from §P3.

- The incoming and outgoing photon directions $\hat{n}'$ and $\hat{n}$ define the scattering plane. It makes sense to consider Stokes parameters defined wrt to this plane:



$$I \equiv \langle \hat{n}_\parallel \rangle + \langle \hat{n}_\perp \rangle \qquad Q \equiv \langle \hat{n}_\parallel \rangle - \langle \hat{n}_\perp \rangle \tag{7}$$

$$\underline{I_\parallel \equiv \tfrac{1}{2}(I+Q) = \langle \hat{n}_\parallel \rangle} \qquad \underline{I_\perp \equiv \tfrac{1}{2}(I-Q) = \langle \hat{n}_\perp \rangle}$$

where we introduced the Stokes parametrization $I_\parallel, I_\perp$ as an alternative to I, Q.

For the chosen polarization basis we have

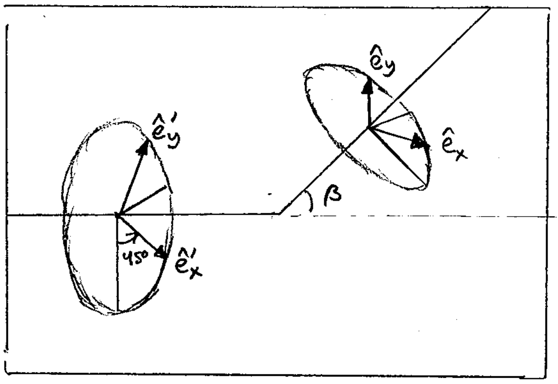
$$\hat{e}_\parallel \cdot \hat{e}_\parallel' = \cos\beta \qquad \hat{e}_\perp \cdot \hat{e}_\parallel' = 0 \tag{8}$$
$$\hat{e}_\parallel \cdot \hat{e}_\perp' = 0 \qquad \hat{e}_\perp \cdot \hat{e}_\perp' = 1$$

- We thus get the contributions to the gain terms of the Stokes parameters $I_\parallel(\vec{q})$, $I_\perp(\vec{q})$ from the incoming photons w momentum $\vec{q}' = q\hat{n}'$, with $\hat{n}' \in d\Omega'$:

$$\frac{dI_\parallel(\vec{q})}{dt} = + n_e \frac{3\sigma_T}{8\pi} \times I_\parallel(\vec{q}')\cdot|\hat{e}_\parallel\cdot\hat{e}_\parallel'|^2 d\Omega' + n_e \frac{3\sigma_T}{8\pi} \times I_\perp(\vec{q}')\,|\hat{e}_\parallel\cdot\hat{e}_\perp'|^2 d\Omega'$$
$$= n_e \frac{3\sigma_T}{8\pi} \times I_\parallel(\vec{q}')\cos^2\beta\, d\Omega', \qquad \text{where } \cos\beta = \hat{n}\cdot\hat{n}' \tag{9}$$
$$\frac{dI_\perp(\vec{q})}{dt} = + n_e \frac{3\sigma_T}{8\pi} \times I_\parallel(\vec{q}')\,|\hat{e}_\perp\cdot\hat{e}_\parallel'|^2 d\Omega' + n_e \frac{3\sigma}{8\pi} \times I_\perp(\vec{q}')\,|\hat{e}_\perp\cdot\hat{e}_\perp'|^2 d\Omega'$$
$$= n_e \frac{3\sigma_T}{8\pi} \times I_\perp(\vec{q}')\,d\Omega'.$$

- We don't get what happens to U or V if we work in this basis. (Choosing a polarization basis, e.g., $\{\hat{e}_\parallel', \hat{e}_\perp'\}$ corresponds to measuring whether the incoming photons have $\hat{e}_\parallel$ or $\hat{e}_\perp$ polarization. After the measurement we have just $n_\parallel$ photons in the $\hat{e}_\parallel$ state and $n_\perp$ photons in the $\hat{e}_\perp$ state and no phase correlations; so this corresponds to the case $U(\vec{q}') = V(\vec{q}') = 0$ for the incoming radiation.

- To get what happens to the Stokes U, consider a linear polarization basis rotated by 45° from the scattering plane.



$$
\begin{aligned}
\hat{e}_x &\equiv \tfrac{1}{\sqrt{2}}(\hat{e}_{\parallel}+\hat{e}_{\perp}) \\
\hat{e}_y &\equiv \tfrac{1}{\sqrt{2}}(-\hat{e}_{\parallel}+\hat{e}_{\perp}) \\
\hat{e}_x' &\equiv \tfrac{1}{\sqrt{2}}(\hat{e}_{\parallel}'+\hat{e}_{\perp}') \\
\hat{e}_y' &\equiv \tfrac{1}{\sqrt{2}}(-\hat{e}_{\parallel}'+\hat{e}_{\perp}')
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
I &\equiv \langle \hat{n}_x\rangle + \langle \hat{n}_y\rangle & U &\equiv \langle \hat{n}_x\rangle - \langle \hat{n}_y\rangle \\
I_x &\equiv \tfrac{1}{2}(I+U) = \langle \hat{n}_x\rangle & I_y &\equiv \tfrac{1}{2}(I-U) = \langle \hat{n}_y\rangle
\end{aligned}
\tag{11}
$$

We got for the gain terms (exercise):

$$
\begin{aligned}
\frac{dI_x(\vec{q})}{dt} &= n_e \frac{3\sigma_T}{16\pi} \times I_x(\vec{q}')\,|\hat{e}_x\cdot\hat{e}_x'|^2 d\Omega' + n_e \frac{3\sigma_T}{16\pi} \times I_y(\vec{q}')\,|\hat{e}_x\cdot\hat{e}_y'|^2 d\Omega' \\
\dots &= n_e \frac{3\sigma_T}{32\pi} \times \left[I(\vec{q}')(1+\cos^2\beta) + U(\vec{q}')\cdot 2\cos\beta \right] d\Omega' \\
\frac{dI_y(\vec{q})}{dt} &= n_e \frac{3\sigma_T}{16\pi} \times I_x(\vec{q}')\,|\hat{e}_y\cdot\hat{e}_x'|^2 d\Omega' + n_e \frac{3\sigma_T}{16\pi} \times I_y(\vec{q}')\,|\hat{e}_y\cdot\hat{e}_y'|^2 d\Omega' \\
\dots &= n_e \frac{3\sigma_T}{32\pi} \times \left[I(\vec{q}')(1+\cos^2\beta) - U(\vec{q}')\cdot 2\cos\beta \right] d\Omega'
\end{aligned}
\tag{12}
$$

$$
\Rightarrow \frac{dU(\vec{q})}{dt} = \frac{dI_x(\vec{q})}{dt} - \frac{dI_y(\vec{q})}{dt} = n_e \frac{3\sigma_T}{8\pi} \times U(\vec{q}')\cos\beta \, d\Omega' \tag{13}
$$

- We get the gain term eqs for I and Q from (9) as

$$\frac{dI(\vec{q})}{dt} = \frac{dI_{\parallel}(\vec{q})}{dt} + \frac{dI_{\perp}(\vec{q})}{dt} = n_e \frac{3\sigma_T}{8\pi} \times \left[I_{\parallel}(\vec{q}\,')\cos^2\beta + I_{\perp}(\vec{q}\,') \right] d\Omega'$$

$$= n_e \frac{3\sigma_T}{16\pi} \times \left[I(\vec{q}\,')(\cos^2\beta + 1) + Q(\vec{q}\,')(\cos^2\beta - 1) \right] d\Omega' \qquad (14)$$

$$\frac{dQ(\vec{q})}{dt} = \frac{dI_{\parallel}(\vec{q})}{dt} - \frac{dI_{\perp}(\vec{q})}{dt} = n_e \frac{3\sigma_T}{8\pi} \times \left[I_{\parallel}(\vec{q}\,')\cos^2\beta - I_{\perp}(\vec{q}\,') \right] d\Omega'$$

$$= n_e \frac{3\sigma_T}{16\pi} \times \left[I(\vec{q}\,')(\cos^2\beta - 1) + Q(\vec{q}\,')(\cos^2\beta + 1) \right] d\Omega'$$

- The problem in the preceding treatment is that the Stokes parameters get defined differently for different incoming and outgoing directions $\hat{n}$ and $\hat{n}'$. Thus we cannot yet integrate over the incoming directions $\hat{n}'$ ($d\Omega'$). First we need to rotate the Stokes parameters to a common basis. This will be done in later sections.

- We thus summarize the results we just obtained as:

$$\frac{d}{dt\,d\Omega'} \begin{bmatrix} I_{\parallel}(\vec{q}) \\ I_{\perp}(\vec{q}) \\ U(\vec{q}) \end{bmatrix} = n_e \frac{3\sigma_T}{8\pi} \begin{bmatrix} \cos^2\beta & & \\ & 1 & \\ & & \cos\beta \end{bmatrix} \begin{bmatrix} I_{\parallel}(\vec{q}\,') \\ I_{\perp}(\vec{q}\,') \\ U(\vec{q}\,') \end{bmatrix} \qquad (15)$$

or

$$\frac{d}{dt\,d\Omega'} \begin{bmatrix} I(\vec{q}) \\ Q(\vec{q}) \\ U(\vec{q}) \end{bmatrix} = n_e \frac{3\sigma_T}{8\pi} \begin{bmatrix} \frac{1}{2}(\cos^2\beta+1) & \frac{1}{2}(\cos^2\beta-1) & 0 \\ \frac{1}{2}(\cos^2\beta-1) & \frac{1}{2}(\cos^2\beta+1) & 0 \\ 0 & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} I(\vec{q}\,') \\ Q(\vec{q}\,') \\ U(\vec{q}\,') \end{bmatrix} \qquad (16)$$

- What happens to Stokes V can be calculated using the helicity basis (exercise). The result is that it appears in the above eq. in the same way as U, i.e., we get a 4×4 matrix w one more diagonal $\cos\beta$ term.

- Thus we see that if we start from unpolarized radiation, $Q=U=V=0$; Thomson scattering produces Q polarization — however, this was just for a particular choice of polarization crd. system applicable to a particular scattering; in other crd. systems this already appears as U polarization. However, the definition of Stokes V is independent of choice of crd. system.

  $\therefore$ Thomson scattering produces linear polarization: $I \to I, Q, U$

  However, we get V only from V $\qquad V \to V$

- For later applications, it will be convenient to have the scattering matrices of (15,16), in yet another Stokes parametrization: I, $Q+iU$, $Q-iU$. We get (exercise):

$$\boxed{\frac{d}{dt\,d\Omega'}\begin{bmatrix} I(\vec{q}) \\ (Q+iU)(\vec{q}) \\ (Q-iU)(\vec{q}) \end{bmatrix} = n_e \frac{\sigma_T}{4\pi}\cdot\frac{3}{4}\underbrace{\begin{bmatrix} \cos^2\beta+1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\sin^2\beta & \frac{1}{2}(\cos\beta+1)^2 & \frac{1}{2}(\cos\beta-1)^2 \\ -\sin^2\beta & \frac{1}{2}(\cos\beta-1)^2 & \frac{1}{2}(\cos\beta+1)^2 \end{bmatrix}}_{\equiv\, S(\beta)}\begin{bmatrix} I(\vec{q}') \\ (Q+iU)(\vec{q}') \\ (Q-iU)(\vec{q}') \end{bmatrix}} \qquad (17)$$

This Stokes parametrization is convenient for crd. rotations, since they transform as

$$\begin{aligned} \tilde{I} &= I \\ \tilde{Q}+i\tilde{U} &= e^{-i2\psi}(Q+iU) \\ \tilde{Q}-i\tilde{U} &= e^{+i2\psi}(Q-iU) \end{aligned} \qquad (18)$$

we will need to use this later, when we transform the Stokes parameters of all photon directions (for a given perturbation Fourier mode) to a common ϑ,φ crd. basis.