Due on Tuesday, December 11th, by 14:00 o'clock. There are no lectures or exercises on December 6th, due to Independence Day.

1. **Linear polarization tensor on flat sky.** Show that the results we got for the components of the polarization tensor in the flat-sky approximation,

$$\begin{aligned} P_{xx} = -P_{yy} &= +\tfrac{1}{2}\left(\partial_x\partial_x A - \partial_x\partial_y B - \partial_y\partial_y A - \partial_y\partial_x B\right) \\ P_{xy} = P_{yx} &= +\tfrac{1}{2}\left(\partial_y\partial_x A - \partial_y\partial_y B + \partial_x\partial_y A + \partial_x\partial_x B\right) \end{aligned}$$

can be written as the single tensor equation

$$P_{ab} = A_{,ab} - \tfrac{1}{2}\delta_{ab}A_{,cc} + \tfrac{1}{2}\left(\epsilon_{cb}B_{,ac} + \epsilon_{ca}B_{,bc}\right),$$

where we are using the Euclidean metric $g_{ab} = \delta_{ab}$ and the components of the Levi-Civita tensor are

$$\epsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

2. **Tensor spherical harmonics for $l = 0$ and $l = 1$.** Show that

$$Y^m_{\ell\ ;ab} - \tfrac{1}{2}g_{ab}Y^{m\ ;c}_{\ell\ \ ;c} = \epsilon^c{}_b Y^m_{\ell;\ ac} + \epsilon^c{}_a Y^m_{\ell;\ bc} = 0$$

for $\ell = 0$ and $\ell = 1$.

3. **Components of the tensor spherical harmonics.** Show that the components of the tensor spherical harmonics (in the θ, ϕ coordinate basis) are

$$\begin{aligned} Y^G_{(\ell m)ab} &= \frac{N_\ell}{2}\begin{bmatrix} W_{\ell m} & X_{\ell m}\sin\theta \\ X_{\ell m}\sin\theta & -W_{\ell m}\sin^2\theta \end{bmatrix} \\ Y^C_{(\ell m)ab} &= \frac{N_\ell}{2}\begin{bmatrix} -X_{\ell m} & W_{\ell m}\sin\theta \\ W_{\ell m}\sin\theta & X_{\ell m}\sin^2\theta \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} W_{\ell m} &\equiv Y^m_{\ell\ ;\theta\theta} - \frac{1}{\sin^2\theta}Y^m_{\ell\ ;\phi\phi} \\ X_{\ell m} &\equiv \frac{2}{\sin\theta}Y^m_{\ell\ ;\theta\phi}. \end{aligned}$$

4. **Polarization correlation functions.** Assuming that the temperature and polarization perturbations were produced by a statistically isotropic parity-conserving Gaussian random process, derive the correlation functions

$$\begin{aligned} C^{UU}(\theta) &= \sum_\ell \frac{2\ell+1}{2\pi}N_\ell^2\left[C_\ell^{GG}G^-_{\ell 2}(\cos\theta) + C_\ell^{CC}G^+_{\ell 2}(\cos\theta)\right] \\ C^{TQ}(\theta) &= \sum_\ell \frac{2\ell+1}{4\pi}N_\ell C_\ell^{TG}P_\ell^2(\cos\theta) \\ C^{QU}(\theta) &= 0. \end{aligned}$$

The functions $G^-_{\ell m}(\cos\theta)$ and $G^+_{\ell m}(\cos\theta)$ are defined in the lecture notes. Note that for the correlation functions the Q and U are defined with respect to a coordinate system, where the "x" -direction corresponds to the great circle connecting the two points whose correlation we are describing. Explain why the QU and TU correlations vanish.