Due on Tuesday, November 20th, by 14:00 o'clock.

1. **Fluid and brightness function description of photons.** Consider a single Fourier mode $\vec{k}$. For scalar perturbations, the lowest multipoles of the collisionless brightness equation are

$$\begin{aligned}\Theta_0^{0\prime} &= -\tfrac{1}{3}k\Theta_1^0 + \Psi' \\ \Theta_1^{0\prime} &= -\tfrac{2}{5}k\Theta_2^0 + k\Theta_0^0 + k\Phi\,.\end{aligned}$$

Derive from these the equations for the photon energy tensor perturbations

$$\begin{aligned}\delta_\gamma' &= -\tfrac{4}{3}kv_\gamma + 4\Psi' \\ v_\gamma' &= \tfrac{1}{4}k\delta_\gamma - \tfrac{1}{6}k\Pi_\gamma + k\Phi\,.\end{aligned}$$

2. **Rotation of the Stokes parameters.** Show that, if the xy-coordinate system is rotated by an angle ϕ, the Stokes parameters transform as

$$\begin{aligned}I' &= I \\ Q' &= \cos 2\phi\; Q + \sin 2\phi\; U \\ U' &= -\sin 2\phi\; Q + \cos 2\phi\; U \\ V' &= V\,.\end{aligned}$$

3. **Polarization ellipse.** For a monochromatic plane wave the electric field vector $\vec{\mathcal{E}}(t)$ draws an ellipse in one oscillation period. Let a and b be the lengths of the major an minor axes of the ellipse, δ the angle between the major axis and the x axis, and $\epsilon \equiv \arctan(b/a)$ ($-\pi/2 \le \delta \le \pi/2$ and $-\pi/2 \le \epsilon \le \pi/2$). Show that

$$\begin{aligned}I &= a^2 + b^2 \\ Q &= I\cos 2\epsilon \cos 2\delta \\ U &= I\cos 2\epsilon \sin 2\delta \\ V &= I \sin 2\epsilon\,.\end{aligned}$$

(Hint: rotation of xy-axes may be helpful.)

4. **Polarization percentage.** We define the polarization percentage as

$$p \equiv \frac{\sqrt{Q^2+U^2+V^2}}{I}\,. \tag{1}$$

Show that for a classical monochromatic plane wave $p = 1$ ($\equiv 100\%$), i.e., that $I^2 = Q^2+U^2+V^2$. Show that for a quasi-monochromatic plane wave $0 \le p \le 1$, i.e., that $0 \le Q^2+U^2+V^2 \le I^2$. (If the general case turns out to be difficult, consider just the case where only the phases $\alpha_x(t)$ and $\alpha_y(t)$ vary in time, but the amplitudes a_x and a_y stay constant.) Show in particular that $p = 0$ is possible.