Due on Tuesday, October 30th, by 14:00 o'clock. There will be no lecture or exercises on November 7th and 8th.

1. **Comoving curvature perturbation.** The comoving gauge (for scalar perturbations) is defined by the conditions $v = B = 0$. This means that the space coordinate lines $x^i = \text{const.}$ coincide with the fluid flow lines, and that the $\eta = \text{const.}$ hypersurface is orthogonal to them. We get to the comoving gauge by the gauge transformation $\xi' = -v$, $\xi^0 = v - B$. Thus $\psi^C = \psi + \mathcal{H}(v - B)$. We define the comoving curvature perturbation as

$$\mathcal{R} \equiv -\psi^C$$

and in terms of Newtonian gauge quantities it is thus

$$\mathcal{R} \;=\; -\psi^N - \mathcal{H}v^N \;=\; -\Psi - \frac{2}{3(1+w)}\left(\mathcal{H}^{-1}\Psi' + \Phi\right) .$$

Using background relations and Einstein equations, show that the evolution equation for $\mathcal{R}$ can be written as

$$\frac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' \;=\; \left(\frac{k}{\mathcal{H}}\right)^2 \left[c_s^2\Psi + \frac{1}{3}(\Psi - \Phi)\right] + \frac{9}{2}c_s^2(1+w)\mathcal{S} .$$

From this we see that for adiabatic perturbations $\mathcal{R} = \text{const.}$ for superhorizon scales ($k \ll \mathcal{H}$). Show also that

$$\frac{2}{3}\mathcal{H}^{-1}\Psi' + \frac{5+3w}{3}\Psi \;=\; -(1+w)\mathcal{R} + 2w\left(\frac{\mathcal{H}}{k}\right)^2 \Pi .$$

2. **Redshift of photon energy.** The photon energy observed by an observer at rest ($x^i = \text{const.}$) in the coordinate system is given by $q \equiv -u_\mu p^\mu$, where $\mathbf{u} = \mathbf{e}_{\hat{0}}$ is the observer 4-velocity and $\mathbf{p}$ is the photon 4-momentum. Derive the photon redshift equation

$$\frac{1}{q}\frac{dq}{d\eta} \;=\; -\mathcal{H} - \tfrac{1}{2}h'_{ij}n^in^j - h'_{0i}n^i + \tfrac{1}{2}h_{00,i}n^i$$

for a general metric perturbation $g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$. The Christoffel symbols are (you don't have to derive these)

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} - \tfrac{1}{2}h'_{00} \\
\Gamma^0_{0i} &= \mathcal{H}h_{0i} - \tfrac{1}{2}h_{00,i} \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + \mathcal{H}\delta_{ij}h_{00} + \mathcal{H}h_{ij} - \tfrac{1}{2}(h_{0i,j} + h_{0j,i}) + \tfrac{1}{2}h'_{ij} \\
\Gamma^i_{00} &= \mathcal{H}h_{0i} + h'_{0i} - \tfrac{1}{2}h_{00,i} \\
\Gamma^i_{0j} &= \mathcal{H}\delta_{ij} - \tfrac{1}{2}(h_{0j,i} - h_{0i,j}) + \tfrac{1}{2}h'_{ij} \\
\Gamma^i_{jk} &= -\mathcal{H}\delta_{jk}h_{0i} + \tfrac{1}{2}h_{ij,k} + \tfrac{1}{2}h_{ik,j} - \tfrac{1}{2}h_{jk,i} .
\end{aligned}$$

A derivation is outlined in the lecture notes. You may follow this outline, but you need to fill in the missing steps.