Signed quasiregular curves

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Liouville type growth results

Theorem (Liuoville's theorem)

Bounded entire functions $\mathbb{C} \to \mathbb{C}$ are constant.

Theorem (n-dimensional Liuoville's theorem)

Bounded quasiregular mappings $\mathbb{R}^n \to \mathbb{R}^n$ are constant.

The n-dimensional Liuoville's theorem follows from the following: there exists $\varepsilon = \varepsilon(n, K) > 0$ so that every K-quasiregular mapping $f \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\lim_{|x| \to \infty} |x|^{-\varepsilon} |f(x)| = 0$$

is constant.

Quasiregular mappings

Definition

A continuous mapping $f\colon M \to N$ between oriented Riemannian n-manifolds is K-quasiregular if $f \in W^{1,n}_{\mathrm{loc}}(M,N)$ and

$$||Df||^n \le KJ_f \text{ a.e. in } M,$$

where $||Df||$ is the operator norm and J_f is the Jacobian determinant.

Note that J_f satisfies $f^* \mathrm{vol}_N = J_f \, \mathrm{vol}_M$.

Cohomological obstructions for quasiregular mappings

Theorem (Bonk-Heinonen, 2001)

Let $f : \mathbb{R}^n \to N$ be a nonconstant K-quasiregular mapping into a closed Riemannian n-manifold. Then

$$\dim H^\ell(N) \leq C(n, K)$$

for every $\ell = 0, \ldots, n$.

Theorem (Prywes, 2019)

Let $f : \mathbb{R}^n \to N$ be a nonconstant quasiregular mapping into a closed Riemannian n-manifold. Then

$$\dim H^\ell(N) \leq \binom{n}{\ell}$$

for every $\ell = 0, \ldots, n$.

Bonk-Heinonen type growth of quasiregular mappings

Theorem (Bonk-Heinonen, 2001)

Let $f : \mathbb{R}^n \to N$ be a nonconstant K-quasiregular mapping, where N is a closed Riemannian n-manifold which is not a rational cohomology sphere. There exists $\varepsilon = \varepsilon(n, K) > 0$ for which

$$\liminf_{r \to \infty} \frac{1}{r^{\varepsilon}} \int_{B^n(r)} J_f > 0.$$

Remark

If N is a closed Riemannian n-manifold which is not a rational cohomology sphere, then there exists $1 \leq \ell \leq n-1$ for which $H^{\ell}(N) \neq 0$.

Higher integrability of quasiregular mappings

Proposition (Prywes, 2019)

Let $f\colon \mathbb{R}^n \to N$ *be a nonconstant* K*-quasiregular mapping into a closed Riemannian n-manifold which is not a rational cohomology sphere. Then there exists* $p = p(n, K, N) > 1$ *and* $C = C(n, K, N) > 0$ *for which*

$$\left(\frac{1}{\left|\frac{1}{2}B\right|} \int_{\frac{1}{2}B} J_f^p \right)^{\frac{1}{p}} \leq C_p \frac{1}{|B|} \int_B J_f$$

for every ball B*.*

Higher integrability of quasiregular mappings

Lemma (Prywes, 2019)

Let N be a Riemannian n-manifold which is closed and not a rational cohomology sphere. Let $\alpha \in \Omega^\ell(N)$ and $\beta \in \Omega^{n-\ell}(N)$ be closed forms satisfying

$$\int_N \alpha \wedge \beta = \int_N \mathrm{vol}_N$$

for some $1 \leq \ell \leq n-1$. Then there exists a smooth positive function h, a smooth partition of unity $\{\lambda_i\}$ and orientation preserving diffeomorphisms $\Phi_i \colon N \to N$ satisfying

$$\mathrm{vol}_N = h \sum_{i=1}^{j} \lambda_i \Phi_i^*(\alpha \wedge \beta).$$

Quasiregular curves

Definition

A smooth n-form $\omega \in \Omega^n(N)$ on a Riemannian m-manifold, $n \leq m$, is an n-volume form if ω is closed and pointwise nonvanishing.

Definition

A continuous mapping $f\colon M \to N$ between oriented Riemannian manifolds, $n = \dim M \leq \dim N$, is a K-quasiregular ω-curve with respect to an n-volume form $\omega \in \Omega^n(N)$ if $f \in W^{1,n}_{\mathrm{loc}}(M, N)$ and

$$(||\omega|| \circ f)\, ||Df||^n \leq K(\star f^*\omega) \text{ a.e. in } M,$$

where $||\omega||$ is the comass norm and the function $(\star f^*\omega)$ satisfies $f^*\omega = (\star f^*\omega)\, \mathrm{vol}_M$.

The algebra $\mathcal{A}_b^n(N)$

Definition

A smooth form $\omega \in \Omega^n(N)$ on a Riemannian manifold N belongs to $\mathcal{A}_b^n(N)$ if

$$\omega = \sum_{i=1}^{j} \varphi_i \alpha_i \wedge \beta_i \tag{1}$$

for some smooth and bounded functions φ_i and smooth, bounded, and closed forms α_i and β_i. The representation (1) is $\mathbb{R}$-linear if all the functions φ_i are constant.

For example, $\mathcal{A}_b^n(\mathbb{R}^m) = \Omega_b^n(\mathbb{R}^m)$.

Signed quasiregular curves

Definition

A quasiregular ω-curve $f\colon M \to N$ is signed with respect to the n-volume form ω if

- $\omega = \sum_{i=1}^{j} \varphi_i \alpha_i \wedge \beta_i \in \mathcal{A}_b^n(N)$ and
- the measurable functions $(\star f^*(\alpha_i \wedge \beta_i))$ do not change sign.

Example

A holomorphic curve $h = (h_1, \ldots, h_k)\colon \mathbb{C} \to \mathbb{C}^k$ is a 1-quasiregular with respect to the standard symplectic form $dx_1 \wedge dy_1 + \cdots + dx_k \wedge dy_k \in \mathcal{A}_b^2(\mathbb{C}^k)$ and the functions $(\star h^*(dx_i \wedge dy_i)) = J_{h_i}$ are nonnegative.

Quasiregular mappings are signed

Let N be a Riemannian n-manifold which is closed and not a rational cohomology sphere. Let $c \in H^\ell(N)$ be nontrivial, $1 \le \ell \le n-1$, and let ξ_c be the harmonic representative of c.
There exists a smooth positive function h, a smooth partition of unity $\{\lambda_i\}$ and orientation preserving diffeomorphisms $\Phi_i \colon N \to N$ satisfying

$$\mathrm{vol}_N = \sum_{i=1}^{j} h\lambda_i \Phi_i^*(\xi_c \wedge \star\xi_c) \in \mathcal{A}_b^n(N).$$

Let $f \colon M \to N$ be a quasiregular mapping. Then f is signed with respect to vol_N since

$$f^*(\Phi_i^*(\xi_c \wedge \star\xi_c)) = (||\xi_c||^2 \circ \Phi_i \circ f)(J_{\Phi_i} \circ f) J_f \,\mathrm{vol}_M \,.$$

Bonk-Heinonen type growth of signed quasiregular curves

Theorem

Let $f\colon \mathbb{R}^n \to N$ be a nonconstant K-quasiregular ω-curve, where N is a Riemannian manifold and $\omega \in \mathcal{A}^n_b(N)$ in an n-volume form satisfying $\inf_N ||\omega|| > 0$. If f is signed or ω has an $\mathbb{R}$-linear representation, then there exists $\varepsilon = \varepsilon(n, K, \omega) > 0$ for which

$$\liminf_{r\to\infty} \frac{1}{r^\varepsilon} \int_{B^n(r)} f^*\omega > 0. \tag{2}$$

If a quasiregular ω-curve $f\colon \mathbb{R}^n \to N$ satisfies (2), we say that f has fast growth.

Some corollaries

Corollary

A nonconstant signed quasiregular ω-curve $f : \mathbb{R}^n \to N$ into a closed Riemannian manifold N has fast growth.

Corollary

A nonconstant quasiregular ω-curve $f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to a nonzero covector $\omega \in \Lambda^n(\mathbb{R}^m)$ has fast growth.

Corollary (Bonk-Heinonen)

If N is closed and not a rational cohomology sphere, then every nonconstant quasiregular mapping $f : \mathbb{R}^n \to N$ has fast growth.

Method of proof (part 1)

Theorem

Let $f : \mathbb{R}^n \to N$ be a nonconstant K-quasiregular ω-curve with respect to an n-volume form $\omega \in \Omega^n(N)$. If there exists $p > 1$ and $C_p > 0$ for which $(\star f^*\omega)$ satisfies

$$\left(\frac{1}{|B^n(\frac{r}{2})|} \int_{B^n(\frac{r}{2})} (\star f^*\omega)^p \right)^{\frac{1}{p}} \leq C_p \frac{1}{|B^n(r)|} \int_{B^n(r)} f^*\omega$$

for every $r > 0$, then there exists $\varepsilon = \varepsilon(n, p) > 0$ for which

$$\liminf_{r\to\infty} \frac{1}{r^\varepsilon} \int_{B^n(r)} f^*\omega > 0.$$

Method of proof (part 2)

Theorem

Let $f : \mathbb{R}^n \to N$ be a nonconstant K-quasiregular ω-curve, where N is a Riemannian manifold and $\omega \in \mathcal{A}_b^n(N)$ in an n-volume form satisfying $\inf_N ||\omega|| > 0$. If f is signed or ω has an $\mathbb{R}$-linear representation, then there exists $p = p(n, K, \omega) > 1$ and $C = C(n, K, \omega) > 0$ for which

$$\left(\frac{1}{\left|\frac{1}{2}B\right|} \int_{\frac{1}{2}B} (\star f^* \omega)^p \right)^{\frac{1}{p}} \leq C_p \frac{1}{|B|} \int_B f^* \omega$$

for every ball B.

As a consequence, $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n, N)$ for $q = np > n$.

Proof of part 1

The estimate

$$\left(\frac{1}{\left|B^n(\frac{r}{2})\right|}\int_{B^n(\frac{r}{2})}(\star f^*\omega)^p\right)^{\frac{1}{p}} \leq C_p\frac{1}{|B^n(r)|}\int_{B^n(r)} f^*\omega,$$

implies

$$C_p^{-1}\left|B^n(1)\right|^{1-\frac{1}{p}} 2^{\frac{n}{p}}\left(\int_{B^n(\frac{r}{2})}(\star f^*\omega)^p\right)^{\frac{1}{p}} \leq \frac{1}{r^{\varepsilon}}\int_{B^n(r)} f^*\omega$$

for $\varepsilon = n(1-1/p) > 0$.

Since f is nonconstant and satisfies $(||\omega|| \circ f)\,||Df||^n \leq K(\star f^*\omega)$ a.e., the function $(\star f^*\omega)$ is nonnegative a.e. and nonzero in some set with positive measure.

Proof of part 1

Let $t > 0$ be so that

$$\int_{B^n(\frac{t}{2})} (\star f^*\omega)^p > 0.$$

Then

$$\frac{1}{r^\varepsilon} \int_{B^n(r)} f^*\omega \geq C \left(\int_{B^n(\frac{r}{2})} (\star f^*\omega)^p \right)^{\frac{1}{p}} \geq C \left(\int_{B^n(\frac{t}{2})} (\star f^*\omega)^p \right)^{\frac{1}{p}}$$

for $r \geq t$ and hence

$$\liminf_{r\to\infty} \frac{1}{r^\varepsilon} \int_{B^n(r)} f^*\omega \geq C \left(\int_{B^n(\frac{t}{2})} (\star f^*\omega)^p \right)^{\frac{1}{p}} > 0.$$

Some steps in the proof of part 2

- Choose a suitable representation $\omega = \sum_{i=1}^{j} \varphi_i \alpha_i \wedge \beta_i \in \mathcal{A}_b^n(N)$
- Choose suitable forms τ_i satisfying $d\tau_i = f^*\alpha_i$
- Let $\psi \in C_c^\infty(B)$ be the standard bump function
- Then

$$\int_{\frac{1}{2}B} f^*\omega \leq \int_B \psi f^*\omega = \sum_{i=1}^{j} \int_B \psi(\varphi_i \circ f) f^*(\alpha_i \wedge \beta_i)$$

Some steps in the proof of part 2

- If $(\star f^*(\alpha_i \wedge \beta_i)) \geq 0$, then

$$\begin{aligned}\int_B \psi(\varphi_i \circ f) f^*(\alpha_i \wedge \beta_i) &\leq ||\varphi_i||_\infty \int_B \psi f^*(\alpha_i \wedge \beta_i) \\ &= ||\varphi_i||_\infty \left| \int_B \psi f^*(\alpha_i \wedge \beta_i) \right|\end{aligned}$$

- If $(\star f^*(\alpha_i \wedge \beta_i)) \leq 0$, then

$$\begin{aligned}\int_B \psi(\varphi_i \circ f) f^*(\alpha_i \wedge \beta_i) &= \int_B \psi(-\varphi_i \circ f)(-f^*(\alpha_i \wedge \beta_i)) \\ &\leq ||\varphi_i||_\infty \int_B \psi(-f^*(\alpha_i \wedge \beta_i)) \\ &= ||\varphi_i||_\infty \left| \int_B \psi f^*(\alpha_i \wedge \beta_i) \right|\end{aligned}$$

Some steps in the proof of part 2

- If $\varphi_i \equiv c_i$, then

$$\int_B \psi(\varphi_i \circ f) f^*(\alpha_i \wedge \beta_i) = c_i \int_B \psi f^*(\alpha_i \wedge \beta_i)$$
$$\leq ||\varphi_i||_\infty \left| \int_B \psi f^*(\alpha_i \wedge \beta_i) \right|$$

- Suffices to estimate

$$\left| \int_B \psi f^*(\alpha_i \wedge \beta_i) \right| = \left| \int_B d\tau_i \wedge (\psi f^* \beta_i) \right| = \left| \int_B \tau_i \wedge d\psi \wedge f^* \beta_i \right|$$
$$\leq C(n) \int_B |\tau_i| \, |d\psi| \, |f^* \beta_i|$$

- ...

Mattila-Rickman type equidistribution for quasiregular curves

Theorem

Let $f : \mathbb{R}^n \to N$ be a nonconstant quasiregular ω_0-curve into a closed Riemannian manifold N. Suppose that the function $r \mapsto \int_{B^n(r)} f^*\omega_0$ is unbounded. Then, for every $\omega \in \Omega^n(N)$ in the de Rham cohomology class of ω_0,

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{\int_{B^n(r)} f^*\omega}{\int_{B^n(r)} f^*\omega_0} = 1,$$

where E has finite logarithmic measure.

Cohomological value distribution for signed quasiregular curves

Theorem

Let $f\colon \mathbb{R}^n \to N$ be a nonconstant signed quasiregular ω_0-curve into a closed Riemannian manifold N. Then, for every $\omega \in \Omega^n(N)$ in the de Rham cohomology class of ω_0,

$$\liminf_{\substack{r\to\infty \\ r\notin E}} \frac{1}{r^\varepsilon} \int_{B^n(r)} f^*\omega > 0,$$

where $\varepsilon > 0$ and E has finite logarithmic measure. In particular, if $(\star f^*\omega) \geq 0$ a.e., then

$$\liminf_{r\to\infty} \frac{1}{r^\varepsilon} \int_{B^n(r)} f^*\omega > 0.$$

Family of examples

Let $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be the 3-dimensional torus and let $\mathrm{pr}_1, \mathrm{pr}_2 \colon T^3 \to \mathbb{S}^1$ be the first and second projections. Let also

$$\omega = \mathrm{pr}_1^* \mathrm{vol}_{\mathbb{S}^1} \wedge \mathrm{pr}_2^* \mathrm{vol}_{\mathbb{S}^1} \in \Omega^2(T^3).$$

Then

- ω is a 2-volume form on T^3,
- $||\omega|| = 1$, and
- ω has an $\mathbb{R}$-linear representation in $\mathcal{A}_b^2(T^3)$.

Let $\pi \colon \mathbb{R}^3 \to T^3$ be the cover map $(x, y, z) \mapsto (e^{2\pi i x}, e^{2\pi i y}, e^{2\pi i z})$. Note that $\pi^*\omega = dx \wedge dy \in \Omega^2(\mathbb{R}^3)$.

Family of examples

Let $a = (a_1, a_2) \in \mathbb{R}^2$ and let $L_a \colon \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping $(x, y) \mapsto (x, y, a_1 x + a_2 y)$. Let $f_a \colon \mathbb{R}^2 \to T^3$ be the composed mapping $f_a = \pi \circ L_a$. Then f_a is a $(1 + |a|)^2$-quasiregular ω-curve since

$$||Df_a||^2 = ||D(\pi \circ L_a)||^2 = ||((D\pi) \circ L_a)DL_a||^2 = ||DL_a||^2 \leq (1 + |a|)^2$$

and

$$f_a^*\omega = (\pi \circ L_a)^*\omega = L_a^*(dx \wedge dy) = \mathrm{vol}_{\mathbb{R}^2} .$$

Result: The image $f_a(\mathbb{R}^2) \subset T^3$ is nowhere dense if $a \in \mathbb{Q}^2$ and dense if $a \in (\mathbb{R} \setminus \mathbb{Q})^2$.
Reason: The set $\{e^{2\pi i k\theta} \colon k \in \mathbb{Z}\} \subset \mathbb{S}^1$ is finite if $\theta \in \mathbb{Q}$ and dense if $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

Thank you!