Quasiregular Curves of Small Distortion in Product Manifolds

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Quasiregular mappings

Definition

A continuous mapping $f\colon M \to N$ between oriented Riemannian n-manifolds is K-quasiregular if $f \in W^{1,n}_{\mathrm{loc}}(M, N)$ and

$$||Df||^n \leq K J_f \text{ a.e. in } M,$$

where $||Df||$ is the operator norm and J_f is the Jacobian determinant.

Note that J_f satisfies $f^* \mathrm{vol}_N = J_f \, \mathrm{vol}_M$.

Quasiregular curves

Definition

A smooth n-form $\omega \in \Omega^n(N)$ on a Riemannian m-manifold, $n \leq m$, is an n-volume form if ω is closed and pointwise non-vanishing.

Definition

A continuous mapping $F\colon M \to N$ between oriented Riemannian manifolds, $n = \dim M \leq \dim N$, is a K-quasiregular ω-curve with respect to an n-volume form $\omega \in \Omega^n(N)$ if $F \in W^{1,n}_{\mathrm{loc}}(M, N)$ and

$$(||\omega|| \circ F)\, ||DF||^n \leq K(\star F^*\omega) \text{ a.e. in } M,$$

where $||\omega||$ is the comass norm and the function $(\star F^*\omega)$ satisfies $F^*\omega = (\star F^*\omega)\,\mathrm{vol}_M$.

A product manifold $N = N_1 \times \cdots \times N_k$, where each factor N_i is an oriented Riemannian n-manifold, carries an n-form

$$\mathrm{vol}_N^{\times} = \sum_{i=1}^{k} \pi_i^* \mathrm{vol}_{N_i} \in \Omega^n(N),$$

where each $\pi_i \colon N \to N_i$ is a projection $(p_1, \ldots, p_k) \mapsto p_i$.

Theorem

Let M be an oriented and connected Riemannian n-manifold for $n \geq 3$ and let $N = N_1 \times \cdots \times N_k$ be a product of oriented Riemannian n-manifolds. Let also $F = (f_1, \ldots, f_k)\colon M \to N$ be a non-constant 1-quasiregular $\mathrm{vol}_N^{\times}$-curve. Then there exists an index i_0 so that

- *the coordinate map $f_{i_0}\colon M \to N_{i_0}$ is conformal and*
- *the coordinate maps $f_i\colon M \to N_i$, $i \neq i_0$, are constant.*

Theorem

Let M be an oriented and connected Riemannian n-manifold for $n \geq 3$ and let $N = N_1 \times \cdots \times N_k$ be a product of oriented Riemannian n-manifolds. Then there exists $\varepsilon = \varepsilon(n, k) > 0$ such that each non-constant $(1+\varepsilon)$-quasiregular $\mathrm{vol}_N^{\times}$-curve $F = (f_1, \ldots, f_k)\colon M \to N$ has a unique coordinate map $f_{i_0}\colon M \to N_{i_0}$ which is a quasiregular local homeomorphism.

Theorem

Let $n \geq 3$. Then there exists $k \in \mathbb{N}$ and a non-constant quasiregular $\mathrm{vol}^{\times}_{(\mathbb{R}^n)^k}$-curve $F\colon \mathbb{R}^n \to (\mathbb{R}^n)^k$ which is constant in the lower half-space.

Theorem (Rosay 2010)

Let $K > 1$ and $k \geq 2$. Then there exists a non-constant K-quasiregular $\mathrm{vol}^{\times}_{\mathbb{C}^k}$-curve $F\colon \mathbb{C} \to \mathbb{C}^k$ which is not discrete.

Thank you!