Categoricity in homogeneous complete metric spaces

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Abstract

We introduce a new approach to the model theory of metric structures by defining the notion of a metric abstract elementary class (MAEC) closely resembling the notion of an abstract elementary class. Further we define the framework of a homogeneous MAEC were we additionally assume the existence of arbitrarily large models, joint embedding, amalgamation, homogeneity and a property which we call the perturbation property. We also assume that the Löwenheim-Skolem number, which in this setting refers to the density character of the set instead of the cardinality, is $\aleph_0$. In these settings we prove an analogue of Morley's categoricity transfer theorem. We also give concrete examples of homogeneous MAECs.

1 Introduction

The application of model theory to structures from analysis can be considered to have started in the mid-sixties with the introduction of Banach space ultrapowers by Bretagnolle, Dacunha-Castelle and Krivine in [BDCK66] and [DCK72] and nonstandard hulls by Luxemburg in [Lux69]. In 1981 Krivine and Maurey [KM81] introduced the notion of a stable Banach space inspired by the model theoretic notion.

In [Hen75] Henson introduces a special first order language designed to express when two Banach spaces have isometrically isomorphic nonstandard hulls. The language of positive bounded formulas is introduced in [Hen76] and its model theory is studied extensively in [HI02] by Henson and Iovino. Iovino proves a Lindström-type maximality theorem for it in [Iov01]. In [Iov99a] and [Iov99b] Iovino introduces a notion of stability based on density characters for Banach spaces, develops a notion of forking and proves a stability spectrum result. He also shows that his notion of stability implies the stability defined by Krivine and Maurey in [KM81]. Shelah and Usvyatsov have proved an analogue of Morley's categoricity transfer theorem in this setting and the proof will appear in [SU].

Another approach to metric structures is Ben-Yaacov's notion of compact abstract theories or cats. These were introduced in [BY03] and closely resemble Shelah's Kind II (together with Assumption III) in [She75]. In [BY05] Ben-Yaacov proves an analogue of Morley's theorem for cats.

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Although the general framework of compact abstract theories is more general than that of positive bounded formulas, for metric structures the frameworks are equivalent. The newest approach, which is equivalent to the previous two, is continuous logic. It is based on Chang's and Keisler's work from the 1960's [CK66], but has some crucial differences. One main difference is that when Chang and Keisler allowed any compact Hausdorff space X as a set of truth values, the new approach, introduced in [BYU], fixes $X = [0, 1]$. The advantage of this approach over Henson's logic is that it directly generalizes first order logic and avoids the trouble of approximate satisfaction by having the approximations built into the formulas.

All three approaches mentioned above provide a compact setting for the development of metric model theory. A more general approach is presented by Buechler and Lessmann in [BL03] where they develop forking theory in the framework of simple homogeneous models and provide strongly homogeneous Hilbert spaces as examples of simple structures. The approach is further developed by Buechler and Berenstein in [BB04] where they consider expansions of Hilbert spaces and show that the structures considered are simple stable and have built-in canonical bases.

In this paper we introduce a new approach to the model theory of metric structures. In addition to abandoning compactness we choose not to use any specific language but work in an environment very similar to that of *abstract elementary classes (AEC)*. These were introduced by Shelah in [She87] as a general foundation for model-theoretic studies of non-elementary classes. Our modification of the concept, the *metric abstract elementary class*, is a pair $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ where $\mathbb{K}$ is a class of many-sorted models, each sort being a complete metric space. $\preccurlyeq_{\mathbb{K}}$ is a notion of substructure satisfying natural properties satisfied by the elementary substructure relation in first order languages. The main differences between metric abstract elementary classes and AECs are that we consider complete metric spaces, take the completion of unions when considering closedness under $\preccurlyeq_{\mathbb{K}}$-chains and consider the density character instead of cardinality in the Löwenheim-Skolem number. Of course, if the metric is discrete these changes cancel out.

In addition to the demands of a metric abstract elementary class we also assume joint embedding, amalgamation, the existence of arbitrarily large models, homogeneity and a property which we call the perturbation property. It is our substitute for the perturbation lemma of positive bounded formulas (Proposition 5.15 of [HI02]). A class with these additional properties will be called a homogeneous metric abstract elementary class. We also assume the density-Löwenheim-Skolem number mentioned above to be $\aleph_0$. Recently categoricity has been studied extensively in abstract elementary classes and quite a lot of stability theory has been developed for AECs with amalgamation. However, our assumption of homogeneity makes it possible for us to use the results in homogeneous model theory developed in [HS00].

A main motivation for introducing a new approach is that the authors hope that this setting can be developed to allow for the the study of generalized notions of types based on generalized notions of automorphisms e.g. being automorphic up to perturbations or automorphic via a linear homeomorphism. Consider the example of probability algebras with a generic automorphism studied in [BH04] and [BYB]. This example is unsuperstable (i.e. 'unclassifiable') if stability is measured the usual way from the syntactic types. Since the class has the elimination of quantifiers, to get many types it is enough to look at the syntactic types containing atomic formulas only. On the other hand if we switch the notion of type to be that of being automorphic up to perturbations, the class is omega-stable. Now if one wants to capture this notion of type as a syntactic type, drastic changes to formal language is needed (one can not allow even atomic formulas). It is very difficult to see how this is done, but it is not difficult to see how to modify the approach introduced below to capture these generalized notions of types and isomorphisms (and at least some of the basic results can even be proved).

Our framework generalizes positive bounded theories, cats and continuous logic in the sense that the complete models of a theory in any of these settings forms a homogeneous metric abstract elementary class. We prove the following analogue of Morley's categoricity theorem (Theorem 8.6).

Theorem. *Assume $\mathbb{K}$ is κ-categorical for some $\kappa = \kappa^{\aleph_0} > \aleph_1$. Then there is $\xi < \beth_{(2^{\aleph_0})^+}$ such that $\mathbb{K}$ is categorical in all λ satisfying*

(i) $\lambda \geq \min\{\xi, \kappa\}$,

(ii) $\lambda^{\aleph_0} = \lambda$,

(iii) *for all $\zeta < \lambda$, $\zeta^{\aleph_0} < \lambda$.*

By the conditions on the cardinals in the theorem, the result actually holds regardless of if we consider cardinalities or densities when measuring the size of models. The difficulties in improving the result arise when we want to extract more information from having too many types over some large set. Since 'too many' is measured in the density of the type-space we need a way to keep distances when moving down to a smaller parameter set. In a last chapter we solve the problem by adding the assumption of *metric homogeneity* which roughly states that distances of types have finite witnessing parameter sets. With this extra assumption we acquire (Corollary 10.31)

Theorem. *If $\mathbb{K}$ is metricly homogeneous and κ-categorical (with respect to densities) for some uncountable κ. Assume further that either $\kappa > \aleph_1$ or separable $F^{\mathfrak{M}}_{\omega}$-saturated models exist. Then there exists some $\xi < \beth_{\mathfrak{c}^+}$ such that $\mathbb{K}$ is categorical in all $\lambda \geq \min\{\kappa, \xi\}$.*

The precise settings are defined in section 2. In the third section we give the definitions of the metric on the space of types originally introduced in [HI02] and the stability with respect to density characters defined in [Iov99a]. We also introduce a new version of saturation, the **d**-saturation which similarly to the stability notion considers dense sets of types, and investigate its relation to conventional saturation. The fourth section is devoted to splitting and independence. Again we introduce density-versions of both concepts and use these among others to relate Iovino's stability notion to conventional stability. In the fifth section we build Ehrenfeucht-Mostowski models and show that categoricity implies stability. In the sixth section we show how to introduce a first order language in order to set our monster into the settings of [HS00]. The seventh section introduces primary models and proves a dominance theorem for these and is roughly a modification of the corresponding results in [She90] and [HS00]. In the eight section we put together the pieces and prove the main theorem. The ninth chapter gives examples. We show that the class of all Banach spaces fit into our framework and give an example of a categorical class of Banach spaces which are not Hilbert. In the final chapter we add the assumption of metric homogeneity and prove an improved version of the main theorem. We also show that metric homogeneity holds in our categorical example class.

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2 Homogeneous metric structures

We investigate a class $\mathbb{K}$ of complete metric space structures of some fixed, countable signature τ. We work in a many-sorted context where the structures are of the form

$$\mathscr{M} = \langle \mathscr{A}_0, \mathscr{A}_1, \dots, \mathbb{R}, \mathrm{d}_0, \mathrm{d}_1, \dots, c_o, c_1, \dots, R_0, R_1, \dots, F_0, F_1, \dots \rangle,$$

where

(i) each $\mathscr{A}_i$ is a complete metric space with metric d_i (with values in $\mathbb{R}$),

(ii) $\mathbb{R}$ is an isomorphic copy of the ordered field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, \leq)$,

(iii) each c_i is a constant and each R_i a relation,

(iv) each F_i is a function $F_i : \mathscr{B}_0 \times \dots \times \mathscr{B}_m \to \mathscr{B}_{m+1}$ where $\mathscr{B}_j \in \{\mathscr{A}_0, \mathscr{A}_1, \dots, \mathbb{R}\}$.

We will not specify the sorts of the elements we work with but just refer to the elements of some sort of $\mathscr{M}$ as *elements of* $\mathscr{M}$ and call the union of the sorts *the domain of* $\mathscr{M}$ (we assume the sorts are disjoint). This domain is not a metric space, so by the closure of a (possibly many-sorted) subset A of this domain, denoted $\overline{A}$, we mean the union of the sortwise closures. For any non-complete metric space X we will also denote by $\overline{X}$ the metric completion of X. Furthermore, by the density character of $\mathscr{M}$, $|\mathscr{M}|$, we mean the sum of the density characters of its sorts. By $\mathrm{card}(A)$ we denote the cardinality of A.

After setting the assumptions we will obtain a homogeneous monster model $\mathfrak{M}$. After that all models mentioned will be submodels of $\mathfrak{M}$. Until then a model is an element of $\mathbb{K}$ and we write $\mathscr{A}$, $\mathscr{B}$ and so on for these. We will also use $\mathscr{A}$ for the domain of the model $\mathscr{A}$. A, B etc. will be used for sets and a, b etc. for finite sequences of elements. By $a \in A$ we mean $a \in A^{\mathrm{length}(a)}$. Note that a finite tuple $a \in A$ may consist of elements of different sorts.

By an automorphism we will mean an automorphism of $\mathfrak{M}$. We will write $\mathrm{Aut}(\mathfrak{M}/A)$ for the set of automorphisms of $\mathfrak{M}$ fixing A pointwise. Note that an automorphism of a many-sorted structure preserves the sorts of elements.

κ, λ, ξ and ζ will be used for infinite cardinals, α, β, γ, i and j for ordinals. δ is reserved for limit ordinals and m, n, k and l are reserved for natural numbers. We write $\mathfrak{c}$ for $2^{\aleph_0}$.

Definition 2.1. We call a class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ of τ-structures for some fixed signature τ a *metric abstract elementary class, MAEC,* if the following hold:

(i) Both $\mathbb{K}$ and the binary relation $\preccurlyeq_{\mathbb{K}}$ are closed under isomorphism.

(ii) If $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$ then $\mathscr{A}$ is a substructure of $\mathscr{B}$ (i.e. each sort of $\mathscr{A}$ is a substructure of the corresponding sort of $\mathscr{B}$).

(iii) $\preccurlyeq_{\mathbb{K}}$ is a partial order on $\mathbb{K}$.

(iv) If $(\mathscr{A}_i)_{i<\delta}$ is a $\preccurlyeq_{\mathbb{K}}$-increasing chain then

  (a) the functions in τ can be uniquely defined on the completion of $\bigcup_{i<\delta} \mathscr{A}_i$ such that $\overline{\bigcup_{i<\delta} \mathscr{A}_i} \in \mathbb{K}$,

  (b) for each $j < \delta$, $\mathscr{A}_j \preccurlyeq_{\mathbb{K}} \overline{\bigcup_{i<\delta} \mathscr{A}_i}$,

(c) if each $\mathscr{A}_i \preccurlyeq_{\mathbb{K}} \mathscr{B} \in \mathbb{K}$ then $\overline{\bigcup_{i<\delta} \mathscr{A}_i} \preccurlyeq_{\mathbb{K}} \mathscr{B}$.

(v) If $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$, $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ and $\mathscr{A} \subseteq \mathscr{B}$ then $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$.

(vi) There exists a Löwenheim-Skolem number $\mathrm{LS}^{\mathrm{d}}(\mathbb{K})$ such that if $\mathscr{A} \in \mathbb{K}$ and $A \subset \mathscr{A}$ then there is $\mathscr{B} \supseteq A$ such that $|\mathscr{B}| = |A| + \mathrm{LS}^{\mathrm{d}}(\mathbb{K})$ and $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{A}$.

Notation 2.2. Since we will always be working with a fixed $\mathbb{K}$, we will use the shorthand $\preccurlyeq$ for $\preccurlyeq_{\mathbb{K}}$.

This definition has two differences compared to the definition of an abstract elementary class. In (iv) we consider the completion of the union of a chain instead of the union itself and in (vi) the Löwenheim-Skolem number refers to the density of the added part instead of its cardinality. Without these alterations we would have the definition of an abstract elementary class (cf. [She87]).

Note that we do not not put any explicit demands on the functions except the extendability in (iv). When we define a homogeneous metric abstract elementary class in Definition 2.13 we will also require an additional property called perturbation (Definition 2.12). However as an example we can mention that the condition holds if we have a uniform family of functions as defined in Section 8 of [HI02], i.e. the functions are bounded and uniformly continuous on bounded subsets of their domains in a way that is uniform for the family.

Remark 2.3. Since taking the completion of a set in a metric space only includes adding limits to countable sequences, we note that if $(A_i)_{i<\lambda}$ is an increasing $\subseteq$-chain of closed sets in a given complete metric space and $cf(\lambda) \neq \omega$ then

$$\overline{\bigcup_{i<\lambda} A_i} = \bigcup_{i<\lambda} A_i.$$

From now on we assume $(\mathbb{K}, \preccurlyeq)$ is a metric abstract elementary class and list some properties. Except for Definition 2.12 the definitions are the same as for ordinary abstract elementary classes.

Definition 2.4 ($\mathbb{K}$-embedding). If $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ and $f : \mathscr{A} \to \mathscr{B}$ is an embedding such that $f(\mathscr{A}) \preccurlyeq \mathscr{B}$, f is called a $\mathbb{K}$*-embedding*.

Definition 2.5 (Joint embedding property). $\mathbb{K}$ is said to have the *joint embedding property* if for any $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ there are $\mathscr{C} \in \mathbb{K}$ and $\mathbb{K}$-embeddings $f : \mathscr{A} \to \mathscr{C}$ and $g : \mathscr{B} \to \mathscr{C}$.

Remark 2.6. Note that since a MAEC is closed under isomorphism, when applying the joint embedding property we may assume that one of the embeddings is the identity mapping.

Definition 2.7 (Amalgamation property). $\mathbb{K}$ is said to have the *amalgamation property* if whenever $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}$, $\mathscr{A} \preccurlyeq \mathscr{B}$ and $\mathscr{A} \preccurlyeq \mathscr{C}$ then there are $\mathscr{D} \in \mathbb{K}$ and $\mathbb{K}$-embeddings $f : \mathscr{B} \to \mathscr{D}$ and $g : \mathscr{C} \to \mathscr{D}$ such that $f \restriction \mathscr{A} = g \restriction \mathscr{A}$.

In order to define further properties we need the notion of a *type*. We introduce the notion of a *Galois-type* which was used by Shelah in [She87] and [She99] and further investigated and named by Grossberg in [Gro02]. We first define the Galois-type with respect to a given model. Later, when we have constructed a monster model, we will not need the models any more and can also consider types over arbitrary sets. We then give a new definition for Galois-type in Definition 2.11.

Definition 2.8 (Galois-type in a model). For $\mathscr{A},\mathscr{B} \in \mathbb{K}$ and $\{a_i : i < \alpha\} \subset \mathscr{A}$, $\{b_i : i < \alpha\} \subset \mathscr{B}$ we say that $(a_i)_{i<\alpha}$ *and* $(b_i)_{i<\alpha}$ *have the same Galois-type in* $\mathscr{A}$ *and* $\mathscr{B}$ *respectively,*

$$\mathrm{t}^g_{\mathscr{A}}((a_i)_{i<\alpha}/\emptyset) = \mathrm{t}^g_{\mathscr{B}}((b_i)_{i<\alpha}/\emptyset),$$

if there are $\mathscr{C} \in \mathbb{K}$ and $\mathbb{K}$-embeddings $f : \mathscr{A} \to \mathscr{C}$ and $g : \mathscr{B} \to \mathscr{C}$ such that $f(a_i) = g(b_i)$ for every $i < \alpha$.

The amalgamation property ensures that having the same Galois-type is a transitive relation. It also holds that elements have the same Galois-type in a model and its $\preccurlyeq$-extensions and that $\mathbb{K}$-embeddings preserve Galois-types.

Definition 2.9 (Homogeneity). We call $\mathbb{K}$ *homogeneous* if whenever $\mathscr{A},\mathscr{B} \in \mathbb{K}$, $\{a_i : i < \alpha\} \subset \mathscr{A}$, $\{b_i : i < \alpha\} \subset \mathscr{B}$ and for all $n < \omega$, $i_0,\dots,i_{n-1} < \alpha$

$$\mathrm{t}^g_{\mathscr{A}}((a_{i_0},\dots,a_{i_{n-1}})/\emptyset) = \mathrm{t}^g_{\mathscr{B}}((b_{i_0},\dots,b_{i_{n-1}})/\emptyset)$$

then

$$\mathrm{t}^g_{\mathscr{A}}((a_i)_{i<\alpha}/\emptyset) = \mathrm{t}^g_{\mathscr{B}}((b_i)_{i<\alpha}/\emptyset).$$

With the properties defined so far we can construct a homogeneous monster model. We omit the proof which is a modification of the usual Jónsson-Fraïssé construction (see [Jón56], [Jón60]), only considering completions of unions instead of just unions of chains along the construction.

Theorem 2.10. *Let* $(\mathbb{K},\preccurlyeq)$ *be a metric abstract elementary class of* τ*-structures satisfying the joint embedding property, the amalgamation property and homogeneity. Let* $\mu > |\tau| + \aleph_0$*. Then there is* $\mathfrak{M} \in \mathbb{K}$ *such that*

(i) *(μ-universality):* $\mathfrak{M}$ *is* μ*-universal, that is for all* $\mathscr{A} \in \mathbb{K}$ *with* $|\mathscr{A}| < \mu$ *there is a* $\mathbb{K}$*-embedding* $f : \mathscr{A} \to \mathfrak{M}$*.*

(ii) *(μ-homogeneity): If* $(a_i)_{i<\alpha}$ *and* $(b_i)_{i<\alpha}$ *are sequences of elements of* $\mathfrak{M}$ *such that* $\alpha < \mu$ *and for all* $n < \omega$ *and* $i_0,\dots,i_{n-1} < \alpha$

$$\mathrm{t}^g_{\mathfrak{M}}((a_{i_0},\dots,a_{i_{n-1}})/\emptyset) = \mathrm{t}^g_{\mathfrak{M}}((b_{i_0},\dots,b_{i_{n-1}})/\emptyset)$$

then there is an automorphism f *of* $\mathfrak{M}$ *such that* $f(a_i) = b_i$ *for all* $i < \alpha$*.*

Definition 2.11 (Galois-type). We say that $\mathrm{t}^g((a_i)_{i<\alpha}/A) = \mathrm{t}^g((b_i)_{i<\alpha}/A)$ if there is $f \in \mathrm{Aut}(\mathfrak{M}/A)$ such that $f(a_i) = b_i$ for every $i < \alpha$.

Note that for $A = \emptyset$ the above definition coincides with the notion of Galois-type in $\mathfrak{M}$ in Definition 2.8.

Finally we introduce one last property concerning types. The importance of it will become clear in the next section where we define the metric on the space of types. The purpose of it is to replace the Perturbation lemma in [HI02].

Definition 2.12 (Perturbation). Assume $\mathbb{K}$ satisfies the joint embedding property, the amalgamation property and homogeneity, so that $\mathfrak{M}$ can be constructed. Then $\mathbb{K}$ is said to have the *perturbation property* if whenever $A \subset \mathfrak{M}$ and $(b_i)_{i<\omega}$ is a convergent sequence with $b = \lim_{i\to\infty} b_i$ such that $\mathrm{t}^g(b_i/A) = \mathrm{t}^g(b_j/A)$ for all $i, j < \omega$, then $\mathrm{t}^g(b/A) = \mathrm{t}^g(b_i/A)$ for $i < \omega$.

Now we are ready to list the assumptions we will make on $\mathbb{K}$.

Definition 2.13 (Homogeneous metric abstract elementary class). We call a metric abstract elementary class $(\mathbb{K}, \preccurlyeq)$ a *homogeneous MAEC* if it satisfies the following properties:

(i) $\mathbb{K}$ contains arbitrarily large models (i.e. models of arbitrarily large density).

(ii) $\mathbb{K}$ has the joint embedding property 2.5.

(iii) $\mathbb{K}$ has the amalgamation property 2.7

(iv) $\mathbb{K}$ is homogeneous as defined in 2.9.

(v) $\mathbb{K}$ satisfies the perturbation property 2.12.

Assumption 2.14. From now on we assume that $\mathbb{K}$ is a homogeneous MAEC of τ-structures with $\mathrm{LS}^{\mathrm{d}}(\mathbb{K}) = \aleph_0$. Hence we can construct a μ-universal, μ-homogeneous monster model $\mathfrak{M}$ for some μ larger than any cardinality we will encounter and consider only $\preccurlyeq$-submodels of $\mathfrak{M}$.

3 Metric saturation and stability notions

Since we live inside the monster model our space of n-types over A, $S^n(A)$, will be the set

$$S^n(A) = \{\mathrm{t}^g(a/A) : \mathrm{length}(a) = n, a \in \mathfrak{M}\}.$$

It is worth noting that the many-sorted context increases the number of types since we in $S(A) = \bigcup_{n<\omega} S^n(A)$ have types for each finite combination of sorts. However, since there are only countably many sorts this only increases the number by a factor of $\aleph_0$.

For each $A \subset \mathfrak{M}$ and $n < \omega$ we define a metric $\mathbf{d}$ on the space $S^n(A)$ in the same way as Henson and Iovino define it for types in the positive bounded logic in [HI02].

Definition 3.1. If $p, q \in S^n(A)$

$$\mathbf{d}(p, q) = \inf\{\mathrm{d}(b, c) : \mathrm{t}^g(b/A) = p, \mathrm{t}^g(c/A) = q\},$$

where $\mathrm{d}(b, c) = \max_{i<n} \mathrm{d}(b_i, c_i)$.

Remark 3.2. Note that if $p, q \in S(A)$ and a realizes p then for every $\varepsilon > 0$ there is a realization b of q such that $\mathrm{d}(a, b) \leq \mathbf{d}(p, q) + \varepsilon$.

It is easy to see that $\mathbf{d}$ is always a pseudometric. To get a metric we need the Perturbation property 2.12: Assume $p, q \in S(A)$ and $\mathbf{d}(p, q) = 0$. Then for each $n < \omega$ there are elements a_n, b_n of types p and q respectively such that $\mathrm{d}(a_n, b_n) < \frac{1}{n+1}$. By the previous remark we may assume $a_n = a_0$ for all $n < \omega$. Then $(b_n)_{n<\omega}$ is a convergent sequence and by perturbation

$$p = \mathrm{t}^g(a_0/A) = \mathrm{t}^g(\lim_{n\to\infty} b_n/A) = \mathrm{t}^g(b_n/A) = q.$$

Further it is clear that if $p, q \in S^n(B)$ and $A \subseteq B$ then $\mathbf{d}(p, q) \geq \mathbf{d}(p \restriction A, q \restriction A)$.

We will use two notions of stability. One is the traditional notion, the other was introduced by Iovino in [Iov99a] and considers the density of the set of types instead of its cardinality.

Definition 3.3. (i) $\mathbb{K}$ is *ξ-stable* if for all $A \subset \mathfrak{M}$ with $|A| \leq \xi$, $\operatorname{card}(S(A)) \leq \xi$.

(ii) $\mathbb{K}$ is *ξ-stable with respect to* $\mathbf{d}$ (*ξ-$\mathbf{d}$-stable* for short) if for all $A \subset \mathfrak{M}$ with $|A| \leq \xi$, $|S(A)|$, the density of $S(A)$ with respect to $\mathbf{d}$, is $\leq \xi$.

When we further on consider a stable $\mathbb{K}$ we will denote by $\lambda(\mathbb{K})$ the smallest ξ such that $\mathbb{K}$ is ξ-stable.

We define saturation and strong saturation with respect to the monster model. What we call $F^{\mathfrak{M}}_{\lambda}$-saturation was called (D, λ)-homogeneity in [She71].

Definition 3.4. We say that a and b have the same *Lascar strong type* over A, written $\operatorname{Lstp}(a/A) = \operatorname{Lstp}(b/A)$, if $E(a, b)$ holds for any A-invariant equivalence relation E (i.e. $E(a, b)$ iff $E(f(a), f(b))$ for each $f \in \operatorname{Aut}(\mathfrak{M}/A)$) with less than $|\mathfrak{M}|$ equivalence classes.

Definition 3.5. (i) We say that $\mathscr{A}$ is *$F^{\mathfrak{M}}_{\lambda}$-saturated* if for all $A \subseteq \mathscr{A}$ of density $< \lambda$ and all $a \in \mathfrak{M}$, there is $b \in \mathscr{A}$ such that $\mathrm{t}^g(b/A) = \mathrm{t}^g(a/A)$.

(ii) We say that $\mathscr{A}$ is *strongly $F^{\mathfrak{M}}_{\lambda}$-saturated* if for all $A \subseteq \mathscr{A}$ of density $< \lambda$ and all $a \in \mathfrak{M}$, there is $b \in \mathscr{A}$ such that $\operatorname{Lstp}(b/A) = \operatorname{Lstp}(a/A)$.

Note that since automorphisms fixing a dense subset of a given set A must fix the set A, we get the same notion if we consider the cardinality of the parameter set instead of the density character. This will be important when applying results from [HS00] from section 6 onwards.

As with stability we will also use a density version of saturation.

Definition 3.6. We say that a model $\mathscr{A}$ is *densely $F^{\mathfrak{M}}_{\lambda}$-saturated with respect to* $\mathbf{d}$ (*$F^{\mathfrak{M}}_{\lambda}$-$\mathbf{d}$-saturated* for short) if for every $B \subset \mathscr{A}$ with $|B| < \lambda$, $\mathscr{A}$ realizes every type in a $\mathbf{d}$-dense subset of $S(B)$.

Theorem 3.7. *If $\mathscr{A}$ is $F^{\mathfrak{M}}_{\lambda}$-$\mathbf{d}$-saturated then it is $F^{\mathfrak{M}}_{\lambda}$-saturated.*

Proof. Let $A \subset \mathscr{A}$, $|A| < \lambda$ and $a \in \mathfrak{M}$. We wish to realize $\mathrm{t}^g(a/A)$ in $\mathscr{A}$. By induction on $n < \omega$, define b_n and automorphisms g_n, f_n such that

(i) $b_n \in \mathscr{A}$,

(ii) $f_n \in \operatorname{Aut}(\mathfrak{M}/A \cup \{b_m : m < n\})$,

(iii) $g_0 = \mathrm{id}$ and $g_{n+1} = f_n \circ g_n$,

(iv) $\mathbf{d}(\mathrm{t}^g(b_n/A \cup \{b_m : m < n\}), \mathrm{t}^g(g_n(a)/A \cup \{b_m : m < n\})) \leq 2^{-(n+2)}$,

(v) $\mathrm{d}(g_{n+1}(a), b_n) \leq 2^{-(n+1)}$.

By $F^{\mathfrak{M}}_{\lambda}$-$\mathbf{d}$-saturation we can satisfy (i) and (iv) and then by 3.2 we can find a' such that $\mathrm{t}^g(a'/A \cup \{b_m : m < n\}) = \mathrm{t}^g(g_n(a)/A \cup \{b_m : m < n\})$ and $\mathrm{d}(a', b_n) \leq 2^{-(n+1)}$. Hence there is $f_n \in \operatorname{Aut}(\mathfrak{M}/A \cup \{b_m : m < n\})$ mapping $g_n(a)$ to a'. Then (ii) and (v) are satisfied (and (iii) is just a definition).

Now since $f_{n+1} \in \operatorname{Aut}(\mathfrak{M}/A \cup \{b_m : m \leq n\})$ and distances are preserved under automorphisms we get

$$\begin{aligned}
\mathrm{d}(b_{n+1}, b_n) &\leq \mathrm{d}(b_{n+1}, g_{n+2}(a)) + \mathrm{d}(g_{n+2}(a), b_n) \\
&= \mathrm{d}(b_{n+1}, g_{n+2}(a)) + \mathrm{d}(g_{n+1}(a), b_n) \\
&\leq 2^{-(n+2)} + 2^{-(n+1)} \\
&\leq 2^{-n}.
\end{aligned}$$

Thus the sequence $(b_n)_{n<\omega}$ is a Cauchy sequence and we can define $b = \lim_{n\to\infty} b_n$. Since for all n, $g_n \in \mathrm{Aut}(\mathfrak{M}/A)$ we have $\mathrm{t}^g(g_n(a)/A) = \mathrm{t}^g(a/A)$ and by (v) $(b_n)_{n<\omega}$ and $(g_n(a))_{n<\omega}$ have the same limit, so by perturbation

$$\mathrm{t}^g(b/A) = \mathrm{t}^g(\lim_{n\to\infty} b_n/A) = \mathrm{t}^g(\lim_{n\to\infty} g_n(a)/A) = \mathrm{t}^g(g_n(a)/A) = \mathrm{t}^g(a/A).$$

□

We introduce yet another notion of saturation. This is a modification of the notion of *approximate saturation* introduced by Ben-Yaacov in [BY05]. The essential part of Theorem 3.10 is a modification of the proof of [BYU07, Fact 1.5].

Definition 3.8. A model $\mathscr{A}$ is called *approximately $F^{\mathfrak{M}}_{\lambda}$-saturated* if for all $A = \{a_i : i < \xi\} \subseteq \mathscr{A}$ with $\xi < \lambda$, all $a \in \mathfrak{M}$ and $\varepsilon > 0$ there are $b \in \mathscr{A}$ and $A' = \{a'_i : i < \xi\} \subseteq \mathscr{A}$ such that $\mathrm{d}(a_i, a'_i) \leq \varepsilon$ for every $i < \xi$ and

$$\mathrm{t}^g(b/A') = \mathrm{t}^g(a/A').$$

Lemma 3.9. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then there exists a separable approximately $F^{\mathfrak{M}}_{\omega}$-saturated model containing any given separable set A.*

Proof. By $\mathrm{LS}^{\mathrm{d}} = \aleph_0$ and ω-$\mathbf{d}$-stability, construct inductively separable models $\mathscr{A}_n$, $n < \omega$, satisfying

(i) $\mathscr{A}_n \supseteq A$,

(ii) $\mathscr{A}_n \preccurlyeq \mathfrak{M}$,

(iii) $\mathscr{A}_n \subseteq \mathscr{A}_{n+1}$ and $\mathscr{A}_{n+1}$ realizes a dense subset of $S(\mathscr{A}_n)$.

Then we get

$$\mathscr{A}_\omega = \overline{\bigcup_{n<\omega} \mathscr{A}_n} \preccurlyeq \mathfrak{M}.$$

Now if $B \subset \mathscr{A}_\omega$ is finite and $\varepsilon > 0$ there is $B' \subseteq \bigcup_{n<\omega} \mathscr{A}_n$ with $|B'| = |B|$ and $\mathrm{d}(b, b') \leq \varepsilon$ for corresponding elements $b \in B$, $b' \in B'$. Since B' is finite there is $m < \omega$ such that $B' \subset \mathscr{A}_m$. Now for any $a \in \mathfrak{M}$ by a construction similar to that in 3.7 (choose $b_n \in \mathscr{A}_{m+n}$) we see that $\mathrm{t}^g(a/B')$ is realized in $\mathscr{A}_\omega$. □

Theorem 3.10. *If A is (metricly) complete and approximately $F^{\mathfrak{M}}_{\omega}$saturated then for any separable $A^+ \subseteq A$ there is a separable model $\mathscr{A}$ such that $A^+ \subseteq \mathscr{A} \subseteq A$.*

Proof. By the previous lemma there is a separable approximately $F^{\mathfrak{M}}_{\omega}$-saturated model $\mathscr{B} \preccurlyeq \mathfrak{M}$. Choose countable dense sets $A' \subseteq A^+$ and $B' \subseteq \mathscr{B}$ and enumerate them $A' = \{a_n : n < \omega\}$, $B' = \{b_n : n < \omega\}$.

We will construct increasing sequences of finite sets $A_n \subset A$ and $B_n \subset \mathscr{B}$ and automorphisms f_n, g_n such that

(i) $f_n(A_n) \subset \mathscr{B}$,

(ii) $g_n(B_n) \subset A$,

(iii) $A_0 = B_0 = \emptyset$,

(iv) $A_{n+1} = \{a_i : i \leq n\} \cup A_n \cup g_n(B_n)$,

(v) $B_{n+1} = \{b_i : i \leq n\} \cup B_n \cup f_{n+1}(A_{n+1})$,

(vi) for all $c \in A_n$, $\mathrm{d}(c, g_n \circ f_n(c)) \leq 2^{-n}$,

(vii) for all $c \in B_n$, $\mathrm{d}(c, f_{n+1} \circ g_n(c)) \leq 2^{-n}$.

We start with $f_0 = \mathrm{id}_{\mathfrak{M}}$. Then assume A_n, B_n and f_n have been defined and consider $\mathrm{t}^g(f_n^{-1}(B_n)/A_n)$. Since A is approximately $F_\omega^{\mathfrak{M}}$-saturated there are $B_n^*, A_n' \subset A$ such that $\mathrm{d}(a, a') \leq 2^{-(n+1)}$ for corresponding $a \in A$, $a' \in A'$ and

$$\mathrm{t}^g(B_n^*/A_n') = \mathrm{t}^g(f_n^{-1}(B_n)/A_n').$$

We let g' witness this, i.e. $g' \in \mathrm{Aut}(A_n')$ and $g'(f_n^{-1}(B_n)) = B_n^*$. Then we let $g_n = g' \circ f_n^{-1}$, which is an automorphism of $\mathfrak{M}$ mapping B_n to B_n^*. We construct f_{n+1} from g_n in a similar fashion by considering the type $\mathrm{t}^g(g_n^{-1}(A_{n+1})/B_n)$.

To see that (vi) holds let $c \in A_n$. Then there is $c' \in A_n'$ with $\mathrm{d}(c', c) \leq 2^{-(n+1)}$ and since $\mathrm{t}^g(g_n \circ f_n(c)/A_n') = \mathrm{t}^g(g'(c)/A_n') = \mathrm{t}^g(c/A_n')$ we obtain

$$\mathrm{d}(c, g_n \circ f_n(c)) \leq \mathrm{d}(c, c') + \mathrm{d}(c', g_n \circ f_n(c)) = 2\mathrm{d}(c, c') = 2^{-n}.$$

(vii) is obtained similarly.

Next we note that for $c \in A_n$,

$$\begin{aligned} \mathrm{d}(f_{n+1}(c), f_n(c)) &\leq \mathrm{d}(f_{n+1}(c), f_{n+1} \circ g_n \circ f_n(c)) + \mathrm{d}(f_{n+1} \circ g_n \circ f_n(c), f_n(c)) \\ &\leq \mathrm{d}(c, g_n \circ f_n(c)) + 2^{-n} \\ &\leq 2^{-n} + 2^{-n} \\ &= 2^{-n+1}. \end{aligned}$$

Hence the sequence of mappings $f_n \restriction A_n$ converges pointwisely to a mapping $f' : \bigcup_{n<\omega} A_n \to \mathscr{B}$ and since all f_n are automorphisms, by the perturbation property f' is type preserving and hence extends to an automorphism f of $\mathfrak{M}$, which maps $\overline{\bigcup_{n<\omega} A_n}$ into $\mathscr{B}$ (since $\mathscr{B}$ is complete and $f(\bigcup_{n<\omega} A_n) \subset \mathscr{B}$). Similarly we find an automorphism g mapping $\mathscr{B} = \overline{\bigcup_{n<\omega} B_n}$ into A.

For $n < m < \omega$,

$$\begin{aligned} \mathrm{d}(a_n, g \circ f(a_n)) &\leq \mathrm{d}(a_n, g \circ f_m(a_n)) + \mathrm{d}(f_m(a_n), f(a_n)) \\ &\leq \mathrm{d}(a_n, g_m \circ f_m(a_n)) + \mathrm{d}(g_m \circ f_m(a_n), g \circ f_m(a_n)) + 2^{-m+2} \\ &\leq 2^{-m} + 2^{-m+2} + 2^{-m+2} \\ &< 2^{-m+4}. \end{aligned}$$

By letting $m \to \infty$ we see that $g \circ f$ is the identity on $\bigcup_{n<\omega} A_n$ and hence on $\overline{\bigcup_{n<\omega} A_n}$. Similarly $f \circ g$ is the identity on $\mathscr{B}$. Hence $\overline{\bigcup_{n<\omega} A_n}$ is isomorphic to $\mathscr{B}$ and is the required $\mathscr{A}$. □

Corollary 3.11. *Complete approximately $F_\omega^{\mathfrak{M}}$-saturated sets are models.*

Proof. Using Theorem 3.10 as a starting point it is easy to prove by induction on λ that if $\mathscr{B}$ is complete and approximately $F_\omega^{\mathfrak{M}}$-saturated, then for any set $A \subseteq \mathscr{B}$ of density λ there exists a model $\mathscr{A}$ of density λ satisfying $A \subseteq \mathscr{A} \subseteq \mathscr{B}$. Then the claim follows considering $A = \mathscr{B}$. □

4 Splitting and independence

We define splitting with respect to the Galois-types from Definition 2.11.

Definition 4.1. Let $A \subset B$ and $a \in \mathfrak{M}$. We say that $\mathrm{t}^g(a/B)$ *splits over* A if there are $b, c \in B$ such that

$$\mathrm{t}^g(b/A) = \mathrm{t}^g(c/A)$$

but

$$\mathrm{t}^g(b/A \cup a) \neq \mathrm{t}^g(c/A \cup a).$$

Further we define strong splitting based on the previous definition.

Definition 4.2. A type $\mathrm{t}^g(a/B)$ is said to *split strongly over* $A \subset B$ if there are $b, c \in B$ and an infinite sequence I, indiscernible over A, with $b, c \in I$ such that $\mathrm{t}^g(b/A \cup a) \neq \mathrm{t}^g(c/A \cup a)$. Here indiscernible is with respect to Galois-types, i.e. $(a_i)_{i \in J}$ is indiscernible over C if for any order-preserving finite partial mapping $f : J \to J$ there is $F \in \mathrm{Aut}(\mathfrak{M}/C)$ such that $F(a_i) = a_{f(i)}$ for all $i \in \mathrm{dom}(f)$.

Definition 4.3. We denote by $\kappa(\mathbb{K})$ the least cardinal such that there are no a, b_i and c_i for $i < \kappa(\mathbb{K})$ such that $\mathrm{t}^g(a/\bigcup_{j \leq i}(b_j \cup c_j))$ splits strongly over $\bigcup_{j<i}(b_j \cup c_j)$ for each $i < \kappa(\mathbb{K})$.

Again a metric version of splitting will prove itself useful.

Definition 4.4 (ε-splitting). For $\varepsilon > 0$, $a \in \mathfrak{M}$ and $A \subset B$ we say that $\mathrm{t}^g(a/B)$ *ε-splits over* A if there are $b, c \in B$ and $f \in \mathrm{Aut}(\mathfrak{M}/A)$ such that $f(c) = b$ but $\mathbf{d}(\mathrm{t}^g(a/A \cup b), \mathrm{t}^g(f(a)/A \cup b)) \geq \varepsilon$.

Lemma 4.5. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then for all a and B and every $\varepsilon > 0$ there is a finite $A \subset B$ such that $\mathrm{t}^g(a/B)$ does not ε-split over A.*

Proof. Assume towards a contradiction that $\mathrm{t}^g(a/B)$ ε-splits over every finite $A \subset B$. Choose for $i < \omega$, A_i, b_i, c_i and f_i such that

(i) $A_0 = \emptyset$, $A_i \subset A_{i+1}$ and A_i is finite,

(ii) $\mathrm{t}^g(b_i/A_i) = \mathrm{t}^g(c_i/A_i)$, $b_i, c_i \in A_{i+1}$,

(iii) $f_i \in \mathrm{Aut}(\mathfrak{M}/A_i)$, $f_i(c_i) = b_i$,

(iv) $\mathbf{d}(\mathrm{t}^g(a/A_i \cup b_i), \mathrm{t}^g(f_i(a)/A_i \cup b_i)) \geq \varepsilon$.

Next for every $\eta \in {}^{\omega}2$ define automorphisms $F_{\eta \restriction n}$, $n < \omega$ as follows:

(i) $F_{\eta \restriction 0} = \mathrm{id}$,

(ii) $F_{\eta \restriction n+1} = \begin{cases} F_{\eta \restriction n} & \text{if } \eta(n) = 0 \\ F_{\eta \restriction n} \circ f_n & \text{if } \eta(n) = 1. \end{cases}$

Then $F_{\eta \restriction n} \restriction A_n \subseteq F_{\eta \restriction n+1} \restriction A_{n+1}$ and $\bigcup_{n<\omega} F_{\eta \restriction n} \restriction A_n$ is a type-preserving mapping and can be extended to an automorphism F_η. Now note that for all $\eta \in {}^{\omega}2$ and all $n < \omega$, $\mathrm{t}^g(F_\eta(a)/F_{\eta \restriction n}(A_n)) = \mathrm{t}^g(F_{\eta \restriction n}(a)/F_{\eta \restriction n}(A_n))$ since $F_{\eta \restriction n} \circ F_\eta^{-1}$ is an automorphism mapping $F_\eta(a)$ to $F_{\eta \restriction n}(a)$ and fixing $F_{\eta \restriction n}(A_n)$ pointwise.

Let $D = \bigcup\{F_\chi(A_{\mathrm{length}(\chi)}) : \chi \in {}^{<\omega}2\}$. Then $|D| \le \aleph_0$. Let $\eta \ne \nu \in {}^{\omega}2$ and $n = \min\{n : \eta(n) \ne \nu(n)\}$. Without loss of generality we may assume $\eta(n) = 0$. Now

$$D \supset F_{\nu\restriction n+1}(A_n \cup c_n) = F_{\nu\restriction n} \circ f_n(A_n \cup c_n) = F_{\nu\restriction n}(A_n \cup b_n) = F_{\eta\restriction n}(A_n \cup b_n) = F_{\eta\restriction n+1}(A_n \cup b_n)$$

so

$$\begin{aligned}
&\mathbf{d}(\mathrm{t}^g(F_\eta(a)/D), \mathrm{t}^g(F_\nu(a)/D)) \\
&\ge \mathbf{d}(\mathrm{t}^g(F_\eta(a)/F_{\eta\restriction n+1}(A_n \cup b_n)), \mathrm{t}^g(F_\nu(a)/F_{\nu\restriction n+1}(A_n \cup c_n))) \\
&= \mathbf{d}(\mathrm{t}^g(F_{\eta\restriction n+1}(a)/F_{\eta\restriction n+1}(A_n \cup b_n)), \mathrm{t}^g(F_{\nu\restriction n+1}(a)/F_{\nu\restriction n+1}(A_n \cup c_n))) \\
&= \mathbf{d}(\mathrm{t}^g(F_{\eta\restriction n}(a)/F_{\eta\restriction n}(A_n \cup b_n)), \mathrm{t}^g(F_{\eta\restriction n} \circ f_n(a)/F_{\eta\restriction n}(A_n \cup b_n))) \\
&= \mathbf{d}(\mathrm{t}^g(a/A_n \cup b_n), \mathrm{t}^g(f_n(a)/A_n \cup b_n)) \\
&\ge \varepsilon,
\end{aligned}$$

which contradicts the assumption of ω-$\mathbf{d}$-stability. □

Theorem 4.6. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then for all a and B there is a countable $A \subset B$ such that $\mathrm{t}^g(a/B)$ does not split over A.*

Proof. For each $n < \omega$, by Lemma 4.5 choose A_n such that

(i) $\mathrm{t}^g(a/B)$ does not $\frac{1}{n+1}$-split over A_n,

(ii) $A_n \subset A_{n+1}$,

(iii) A_n is finite.

We claim that $A = \bigcup_{n<\omega} A_n$ is as wanted. If $\mathrm{t}^g(a/B)$ splits over A there are $b, c \in B$ such that $\mathrm{t}^g(b/A) = \mathrm{t}^g(c/A)$ but $\mathrm{t}^g(b/A \cup a) \ne \mathrm{t}^g(c/A \cup a)$. Let $f \in \mathrm{Aut}(\mathfrak{M}/A)$, $f(c) = b$. Now we cannot have $\mathrm{t}^g(a/A \cup b) = \mathrm{t}^g(f(a)/A \cup b)$ since then there would exist $g \in \mathrm{Aut}(\mathfrak{M}/A \cup b)$ mapping $f(a)$ to a. But then $g \circ f \in \mathrm{Aut}(\mathfrak{M}/A)$, $(g \circ f)(c) = b$ and $(g \circ f)(a) = a$, so $\mathrm{t}^g(b/A \cup a) = \mathrm{t}^g(c/A \cup a)$, a contradiction. Hence $\mathbf{d}(\mathrm{t}^g(a/A \cup b), \mathrm{t}^g(f(a)/A \cup b)) > 0$ so there is a positive ε such that $\mathrm{t}^g(a/B)$ ε-splits over A and hence over every A_n, a contradiction for $n > \frac{1}{\varepsilon}$. □

Corollary 4.7. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then there is no increasing sequence $(A_i)_{i<\aleph_1}$ and $b \in \mathfrak{M}$ such that $\mathrm{t}^g(b/A_{i+1})$ splits over A_i for every $i < \aleph_1$.*

Proof. If there were such a sequence then $B = \bigcup_{i<\aleph_1} A_i$ and b would contradict Theorem 4.6 since every countable $A \subseteq B$ is contained in some A_i. □

Corollary 4.8. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then $\kappa(\mathbb{K}) \le \aleph_1$.*

Proof. Immediate since strong splitting implies splitting. □

We define independence as in [HS00].

Definition 4.9. (i) We write $a \downarrow_A B$ and say that a is *independent from B over A* if there is $C \subseteq A$ of cardinality $< \kappa(\mathbb{K})$ such that for all $D \supseteq A \cup B$ there is b satisfying

$$\mathrm{t}^g(b/A \cup B) = \mathrm{t}^g(a/A \cup B) \text{ and } \mathrm{t}^g(b/D) \text{ does not split strongly over } C.$$

We write $C \downarrow_A B$ if for all $a \in C$, $a \downarrow_A B$.

(ii) We say that $\mathrm{t}^g(a/A)$ is bounded if $|\{b : \mathrm{t}^g(b/A) = \mathrm{t}^g(a/A)\}| < |\mathfrak{M}|$, otherwise we call $\mathrm{t}^g(a/A)$ unbounded.

In section 6 we will gain access to results from homogeneous model theory which will show that this definition makes sense. However before that we will introduce another notion resembling independence and with its aid study the stability spectrum of ω-$\mathbf{d}$-stable models.

Definition 4.10. (i) We write $a \downarrow^{\varepsilon}_{A} B$ if there is a finite $C \subseteq A$ such that $\mathrm{t}^g(a/A \cup B)$ does not ε-split over C.

(ii) We write $a \downarrow^{0}_{A} B$ and say that a is 0-*independent* from B over A if for all $\varepsilon > 0$, $a \downarrow^{\varepsilon}_{A} B$.

Lemma 4.11. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then for all a and B there is a countable $A \subseteq B$ such that $a \downarrow^{0}_{A} B$.*

Proof. This we get from the proof of Theorem 4.6. For a given a and B, the A given by the theorem is as required. Namely for each $\varepsilon > 0$, any A_n with $n > \frac{1}{\varepsilon}$ is finite, contained in A and such that $\mathrm{t}^g(a/B)$ does not $\frac{1}{n}$-split and hence not ε-split over A_n. □

Lemma 4.12. *For all a, b and B, if $A \subseteq \mathscr{A} \subseteq B$, $\mathrm{t}^g(a/\mathscr{A}) = \mathrm{t}^g(b/\mathscr{A})$, $a \downarrow^{0}_{A} B$, $b \downarrow^{0}_{A} B$ and $\mathscr{A}$ realizes all types over finite subsets of A realized in B, then $\mathrm{t}^g(a/B) = \mathrm{t}^g(b/B)$.*

Proof. Fix a, b, A, $\mathscr{A}$ and B as in the claim. If $\mathrm{t}^g(a/B) \neq \mathrm{t}^g(b/B)$, then by homogeneity there is some (finite) $c \in B$ such that $\mathrm{t}^g(a/c) \neq \mathrm{t}^g(b/c)$. Then $\mathbf{d}(\mathrm{t}^g(a/c), \mathrm{t}^g(b/c)) > 0$. So it is enough to prove that for each $\varepsilon > 0$, and each $c \in B$, $\mathbf{d}(\mathrm{t}^g(a/c), \mathrm{t}^g(b/c)) \leq \varepsilon$.

Fix $\varepsilon > 0$ and $c \in B$. By the assumption, choose a finite $A' \subset A$ such that neither $\mathrm{t}^g(a/B)$ nor $\mathrm{t}^g(b/B)$ $\frac{\varepsilon}{2}$-splits over A'. Then choose $c' \in \mathscr{A}$ and $f \in \mathrm{Aut}(\mathfrak{M}/A')$ such that $f(c) = c'$. Now note that

$$\mathbf{d}(\mathrm{t}^g(a/c), \mathrm{t}^g(b/c)) = \mathbf{d}(\mathrm{t}^g(f(a)/c'), \mathrm{t}^g(f(b)/c')).$$

and by non-$\frac{\varepsilon}{2}$-splitting, $\mathbf{d}(\mathrm{t}^g(a/c'), \mathrm{t}^g(f(a)/c')) \leq \frac{\varepsilon}{2}$ and $\mathbf{d}(\mathrm{t}^g(b/c'), \mathrm{t}^g(f(b)/c')) \leq \frac{\varepsilon}{2}$. So finally

$$\begin{aligned}
&\mathbf{d}(\mathrm{t}^g(a/c), \mathrm{t}^g(b/c)) \\
&= \mathbf{d}(\mathrm{t}^g(f(a)/c'), \mathrm{t}^g(f(b)/c')) \\
&\leq \mathbf{d}(\mathrm{t}^g(f(a)/c'), \mathrm{t}^g(a/c')) + \mathbf{d}(\mathrm{t}^g(a/c'), \mathrm{t}^g(b/c')) + \mathbf{d}(\mathrm{t}^g(b/c'), \mathrm{t}^g(f(b)/c')) \\
&\leq \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

□

Theorem 4.13. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then it is λ-stable (and especially λ-$\mathbf{d}$-stable) for every $\lambda = \lambda^{\aleph_0}$.*

Proof. Let $|B| \leq \lambda$. By ω-$\mathbf{d}$-stability and the assumption $\lambda = \lambda^{\aleph_0}$ we may assume B is $F^{\mathfrak{M}}_{\omega}$-saturated. Now fix $a_i \in \mathfrak{M}$ for $i < \lambda^+$. For each $i < \lambda^+$ choose by Lemma 4.11 a countable $A_i \subset B$ such that $a_i \downarrow^{0}_{A_i} B$. Since there are only $\lambda^{\aleph_0} = \lambda$ countable subsets of B, λ^+ many of the sets A_i are the same and we denote this set by A. Next by a construction similar to that in Lemma 3.9 (and considerations similar to those in the proof of Theorem 3.7) construct a separable model $\mathscr{A} \supset A$ realizing all types over finite subsets of A realized in B (here we use the $F^{\mathfrak{M}}_{\omega}$-saturation of B to find realizations of the types and not just something arbitrarily close). Since $\mathbb{K}$ is ω-$\mathbf{d}$-stable, there are only separably many types over $\mathscr{A}$ and hence $\mathrm{card}(S(\mathscr{A})) \leq 2^{\aleph_0} \leq \lambda$. So λ^+ many tuples a_i have the same type over $\mathscr{A}$ and hence by Lemma 4.12 have the same type over B. □

Finally we show that if $\mathbb{K}$ is ω-$\mathbf{d}$-stable, $\downarrow^0$ satisfies extension over $F^{\mathfrak{M}}_{\aleph_1}$saturated models. In section 6, when we acquire tools for $\downarrow$, we will see that our two independence notions coincide over $F^{\mathfrak{M}}_{\aleph_1}$-saturated models (see theorem 6.4).

Lemma 4.14. *Assume B_n, b_n, for $n < \omega$, are such that*

(i) $B_n \subseteq B_{n+1}$,

(ii) $\mathrm{t}^g(b_{n+1}/B_n) = \mathrm{t}^g(b_n/B_n)$.

Then there is b such that for all $n < \omega$ $\mathrm{t}^g(b/B_n) = \mathrm{t}^g(b_n/B_n)$.

Proof. We define automorphisms F_n, f_n such that

(i) $F_0 = f_0 = \mathrm{id}$, $F_{n+1} = f_{n+1} \circ F_n$,

(ii) $f_{n+1} \restriction F_n(B_n) = \mathrm{id}$,

(iii) $F_{n+1}(b_{n+1}) = b_0$.

Assume F_i and f_i have been defined for $i \leq n$. By the assumption let $g \in \mathrm{Aut}(\mathfrak{M}/B_n)$ such that $g(b_{n+1}) = b_n$. Let $f_{n+1} = F_n \circ g \circ F_n^{-1}$. Then $f_{n+1} \restriction F_n(B_n) = \mathrm{id}$ and

$$F_{n+1}(b_{n+1}) = f_{n+1}(F_n(b_{n+1})) = F_n \circ g \circ F_n^{-1} \circ F_n(b_{n+1}) = F_n \circ g(b_{n+1}) = F_n(b_n) = b_0,$$

the last step by the induction hypothesis.

By (ii) if $m \leq n$ then $F_m \restriction B_m = F_n \restriction B_m$ so by the homogeneity of $\mathfrak{M}$ we can extend $\bigcup_{n<\omega} F_n \restriction B_n$ to an automorphism F. Let $b = F^{-1}(b_0)$. Then for all $n < \omega$, $(F_n^{-1} \circ F) \restriction B_n = (F_n^{-1} \circ F_n) \restriction B_n = \mathrm{id} \restriction B_n$ and $(F_n^{-1} \circ F)(b) = F_n^{-1}(b_0) = b_n$, so $\mathrm{t}^g(b/B_n) = \mathrm{t}^g(b_n/B_n)$ for all $n < \omega$. □

Theorem 4.15. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable. Then $\downarrow^0$ satisfies extension over $F^{\mathfrak{M}}_{\aleph_1}$-saturated models, i.e. if $a \downarrow^0_A \mathscr{B}$, where $A \subseteq \mathscr{B}$ and $\mathscr{B}$ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated, then for all $D \supseteq \mathscr{B}$ there is some b satisfying $\mathrm{t}^g(b/\mathscr{B}) = \mathrm{t}^g(a/\mathscr{B})$ and $b \downarrow^0_A D$.*

Proof. Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable, $a \downarrow^0_A \mathscr{B}$, $A \subseteq \mathscr{B}$ and $\mathscr{B}$ is $F^{\mathfrak{M}}_{\aleph_1}$saturated. By the assumption, for each $n < \omega$, there is a finite $A_n \subseteq A$ such that $\mathrm{t}^g(a/\mathscr{B})$ does not $\frac{1}{n+1}$-split over A_n. Let $A' = \bigcup_{n<\omega} A_n$ and note that then $\mathrm{t}^g(a/\mathscr{B})$ does not split over A'.

Now let $D \supseteq \mathscr{B}$ be given. We wish to find some b satisfying $\mathrm{t}^g(b/\mathscr{B}) = \mathrm{t}^g(a/\mathscr{B})$ and $b \downarrow^0_A D$.

For each $C \subseteq D$ we will define b_C such that

(i) $\mathrm{t}^g(b_C/A') = \mathrm{t}^g(a/A')$,

(ii) $C_1 \subseteq C_2$ implies $\mathrm{t}^g(b_{C_2}/A' \cup C_1) = \mathrm{t}^g(b_{C_1}/A' \cup C_1)$.

We do this by induction on $\mathrm{card}(C)$ so first assume C is finite. By the $F^{\mathfrak{M}}_{\aleph_1}$-saturation of $\mathscr{B}$ choose $C' \in \mathscr{B}$ and $f_C \in \mathrm{Aut}(\mathfrak{M}/A')$ such that $f_C(C') = C$. Then let $b_C = f_C(a)$ which clearly satisfies (i). For (ii) note that if $C_1 \subseteq C_2$ and $\mathrm{t}^g(b_{C_2}/A' \cup C_1) \neq \mathrm{t}^g(b_{C_1}/A' \cup C_1)$ then $f^{-1}_{C_1}(C_1), f^{-1}_{C_2}(C_1) \in \mathscr{B}$ and the sets have the same type over A' but

$$\mathrm{t}^g(f^{-1}_{C_1}(C_1) \cup a/A') = \mathrm{t}^g(C_1 \cup b_{C_1}/A') \neq \mathrm{t}^g(C_1 \cup b_{C_2}/A') = \mathrm{t}^g(f^{-1}_{C_2}(C_1) \cup a/A')$$

so $\mathrm{t}^g(a/\mathscr{B})$ splits over A', a contradiction.

Next let $C \subseteq D$ be infinite and assume we have defined $b_{C'}$ for all $C' \subset D$ of strictly smaller cardinality. We can write C as $\bigcup_{i<\alpha} C_i$ where $\mathrm{card}(C_i) < \mathrm{card}(C)$ and $C_i \subseteq C_j$ for each $i \leq j < \alpha$. By the induction hypothesis $\mathrm{t}^g(b_{C_j}/A' \cup C_i) = \mathrm{t}^g(b_{C_i}/A' \cup C_i)$ for all $i \leq j < \alpha$ so by Lemma 4.14 there is a b_C such that $\mathrm{t}^g(b_C/A' \cup C_i) = \mathrm{t}^g(b_{C_i}/A' \cup C_i)$ for all $i < \alpha$. Then (i) clearly holds but we still need to show that $\mathrm{t}^g(b_C/A' \cup C)$ is independent of the choice of the sequence $(C_i)_{i<\alpha}$. So let $(C'_j)_{j<\beta}$ be another increasing sequence of sets of cardinality strictly smaller than $\mathrm{card}(C)$ and let b'_C realize the type defined using the sequence $(C'_j)_{j<\beta}$. Now for each finite $C^* \subset C$ there are $i < \alpha$ and $j < \beta$ such that $C^* \subset C_i \cap C'_j$. By the construction of b_C and b'_C we have

$$\mathrm{t}^g(b_C/A' \cup C^*) = \mathrm{t}^g(b_{C_i}/A' \cup C^*) \text{ and } \mathrm{t}^g(b'_C/A' \cup C^*) = \mathrm{t}^g(b_{C'_j}/A' \cup C^*)$$

and by the induction hypothesis

$$\mathrm{t}^g(b_{C_i}/A' \cup C^*) = \mathrm{t}^g(b_{C^*}/A' \cup C^*) = \mathrm{t}^g(b_{C'_j}/A' \cup C^*)$$

So we have $\mathrm{t}^g(b_C/A' \cup C^*) = \mathrm{t}^g(b'_C/A' \cup C^*)$ for every finite $C^* \subset C$, hence by homogeneity $\mathrm{t}^g(b_C/A' \cup C) = \mathrm{t}^g(b'_C/A' \cup C)$.

Denote $b = b_D$. Next we show that $\mathrm{t}^g(b/D)$ does not $\frac{1}{n+1}$-split over A_n, which will prove $b \downarrow^0_A D$. For this let $n < \omega$ and note that if $c, d \in D$ and $f \in \mathrm{Aut}(\mathfrak{M}/A_n)$ such that $f(d) = c$ and

$$\mathbf{d}(\mathrm{t}^g(b/A_n \cup c), \mathrm{t}^g(f(b)/A_n \cup c)) \geq \frac{1}{n+1}$$

then there are $c', d' \in \mathscr{B}$ and $f_{cd} \in \mathrm{Aut}(\mathfrak{M}/A')$ such that $f_{cd}(c'd') = cd$. Furthermore since $\mathrm{t}^g(b/A'cd) = \mathrm{t}^g(b_{cd}/A'cd)$ (recall $b = b_D$ and $b_{cd} = f_{cd}(a)$), there is $f' \in \mathrm{Aut}(\mathfrak{M}/A'cd)$, $f'(b_{cd}) = b$. Denote $g = f' \circ f_{cd}$. Now $g^{-1} \circ f \circ g \in \mathrm{Aut}(\mathfrak{M}/A_n)$, $g^{-1} \circ f \circ g(d') = c'$ and

$$\begin{aligned}\mathbf{d}(\mathrm{t}^g(a/A_nc'), \mathrm{t}^g(g^{-1} \circ f \circ g(a)/A_nc')) &= \mathbf{d}(\mathrm{t}^g(g(a)/A_ng(c')), \mathrm{t}^g(f \circ g(a)/A_ng(c')))\\ &= \mathbf{d}(\mathrm{t}^g(b/A_nc), \mathrm{t}^g(f(b)/A_nc))\\ &\geq \tfrac{1}{n+1},\end{aligned}$$

contradicting the assumption that $\mathrm{t}^g(a/\mathscr{B})$ does not $\frac{1}{n+1}$ split over A_n. Hence we have in fact proven that $b \downarrow^0_{A'} D$ which implies that $\mathrm{t}^g(b/D)$ does not split over A'.

Finally we show that $\mathrm{t}^g(b/\mathscr{B}) = \mathrm{t}^g(a/\mathscr{B})$. By homogeneity it is enough to show that $\mathrm{t}^g(b/C) = \mathrm{t}^g(a/C)$ for every finite $C \subset \mathscr{B}$, so fix some finite $C' \subset \mathscr{B}$ and let $f_C \in \mathrm{Aut}(\mathfrak{M}/A')$ and $C \subset D$ be such that $f_C(C') = C$ and $f_C(a) = b_C$. Now since $\mathrm{t}^g(b/D)$ does not split over A', neither does $\mathrm{t}^g(b_{C\cup C'}/A' \cup C \cup C')$. Hence there is $g \in \mathrm{Aut}(\mathfrak{M}/A' \cup b_{C\cup C'})$ such that $g \restriction C' = f_C \restriction C'$. Also there is $h \in \mathrm{Aut}(\mathfrak{M}/A' \cup C)$ such that $h(b_C) = b_{C\cup C'}$ and similarly $h' \in \mathrm{Aut}(\mathfrak{M}/A' \cup C')$ such that $h'(b_{C'}) = b_{C\cup C'}$. But then $f^{-1}_C \circ h^{-1} \circ g \circ h' \in \mathrm{Aut}(\mathfrak{M}/A' \cup C')$ and

$$\begin{aligned}f^{-1}_C \circ h^{-1} \circ g \circ h'(b_{C'}) &= f^{-1}_C \circ h^{-1} \circ g(b_{C\cup C'})\\ &= f^{-1}_C \circ h^{-1}(b_{C\cup C'})\\ &= f^{-1}_C(b_C)\\ &= a.\end{aligned}$$

Hence

$$t^g(b/A' \cup C') = t^g(b_{C'}/A' \cup C') = t^g(a/A' \cup C')$$

and since C' was arbitrary

$$t^g(b/\mathscr{B}) = t^g(a/\mathscr{B}).$$

□

5 Skolem functions and Ehrenfeucht-Mostowski models

In [She87] Shelah shows that abstract elementary classes are so called PC-classes by introducing a sort of Skolem functions. A similar construction can be done for metric abstract elementary classes with the difference that our Löwenheim-Skolem axiom does not allow us to list all the elements of a model but only a dense subset of it with Skolem functions. Thus we obtain the following:

Fact 5.1. *Let $(\mathbb{K}, \preccurlyeq)$ be a metric abstract elementary class of τ-structures with $|\tau| + \mathrm{LS}^{\mathrm{d}}(\mathbb{K}) \leq \aleph_0$. Then for each $\mathscr{A} \in \mathbb{K}$ we can define an expansion $\mathscr{A}^*$ with Skolem functions F_n^k for $k, n < \omega$ such that*

(i) *if $A \subseteq \mathscr{A}^*$ and A is closed under the functions F_n^k then $\overline{A} \restriction \tau \in \mathbb{K}$ and $\overline{A} \restriction \tau \preccurlyeq \mathscr{A}$,*

(ii) *for all $a \in \mathscr{A}$ $A_a = \{(F_n^{\mathrm{length}(a)})^{\mathscr{A}^*}(a) : n < \omega\}$ is such that*

 (a) *$\overline{A_a} \restriction \tau \in \mathbb{K}$ and $\overline{A_a} \restriction \tau \preccurlyeq \mathscr{A}$,*

 (b) *if $b \subseteq a$ (as sets) then $b \in A_b \subseteq A_a$.*

Notation 5.2. For $A \subseteq \mathscr{A}^*$, $\mathrm{SH}(A)$ will denote the closure of A under the Skolem functions.

Now we will use the Skolem functions to build Ehrenfeucht-Mostowski models. $\mathfrak{M}^*$ will denote the model we get by introducing Skolem functions into the monster model $\mathfrak{M}$ and τ^* will denote the extension of the signature τ of $\mathbb{K}$.

Definition 5.3 ($*$-type). For $(a_i)_{i<\alpha}, (b_i)_{i<\alpha}, A \subset \mathscr{A}^*$ we write

$$t^*((a_i)_{i<\alpha}/A) = t^*((b_i)_{i<\alpha}/A)$$

if for all atomic formulas φ of signature τ^* with parameters from A and all $n < \omega$, $i_0, \dots, i_{n-1} < \alpha$

$$\mathscr{A}^* \models \varphi(a_{i_0}, \dots, a_{i_{n-1}}) \text{ if and only if } \mathscr{A}^* \models \varphi(b_{i_0}, \dots, b_{i_{n-1}}).$$

Lemma 5.4. *If $t^*(a/A) = t^*(b/A)$ then $t^g(a/A) = t^g(b/A)$.*

Proof. Assume $t^*(a/A) = t^*(b/A)$. Then $a \cup A$ and $b \cup A$ have isomorphic Skolem hulls, the closures of which are $\preccurlyeq$-submodels of $\mathfrak{M}$. By the homogeneity of $\mathfrak{M}$ there hence is an automorphism of $\mathfrak{M}$ mapping the first of these models onto the second and hence mapping a to b and fixing A pointwise. □

Definition 5.5. Let $(I,<)$ be a linear ordering. We call $(a_i)_{i\in I}$ a $*$-n*-order-indiscernible sequence over* A if for $i_0<\cdots<i_{n-1}\in I$, $j_0<\cdots<j_{n-1}\in I$

$$\mathrm{t}^*(\langle a_{i_0},\dots a_{i_{n-1}}\rangle/A)=\mathrm{t}^*(\langle a_{j_0},\dots a_{j_{n-1}}\rangle/A).$$

$(a_i)_{i\in I}$ is $*$*-order-indiscernible over* A if it is $*$-n-order-indiscernible over A for all $n<\omega$. If $A=\emptyset$ we omit it.

By an application of the Erdős-Rado theorem we obtain:

Fact 5.6. *Assume τ^* is as above and countable, and $A\subset\mathfrak{M}$ is countable. If $I=(a_i)_{i<\beth_{\mathfrak{c}}^+}$ then there are infinite $I_n\subset I$ for all $n<\omega$ such that each I_n is $*$-n-order-indiscernible over A and for $m\geq n$, for $a_{i_0},\dots a_{i_{n-1}}\in I_m$, $b_{j_0},\dots b_{j_{n-1}}\in I_n$ with $i_0<i_1<\cdots<i_{n-1}$, $j_0<j_1<\cdots<j_{n-1}$*

$$\mathrm{t}^*(\langle a_{i_0},\dots,a_{i_{n-1}}\rangle/A)=\mathrm{t}^*(\langle b_{j_0},\dots,b_{j_{n-1}}\rangle/A).$$

Theorem 5.7. *Let τ be countable. For every linear order $(I,<)$ there is a model $EM(I)\in\mathbb{K}$ of density* $\mathrm{card}(I)+\aleph_0$*, and if I is a well-order, it realizes only a separable set of types over a countable parameter set.*

Proof. Fix a model $\mathscr{A}^*$ as in 5.1, for $\mathscr{A}\in\mathbb{K}$, containing sequences I_n $(n<\omega)$ as in the previous lemma such that each I_n is $*$-n-order-indiscernible (over $\emptyset$). Consider the set

$$D=\{t(x_0,\dots,x_{n-1}):t\text{ a }\tau^*\text{-term},x_0,\dots,x_{n-1}\in I,n<\omega\}.$$

Here we consider the notation $t(x_0,\dots,x_{n-1})$ to mean that the variables of t are among $x_0,\dots,x_{n-1}$ and hence that

$$\begin{aligned}&\{t(x_0,\dots,x_{n-1}):t\text{ a }\tau^*\text{-term}\}\\=&\{t(y_0,\dots,y_{m-1}):t\text{ a }\tau^*\text{-term and }\langle y_0,\dots,y_{m-1}\rangle\text{ a subsequence of }\langle x_0,\dots,x_{n-1}\rangle\}.\end{aligned}$$

We then let $\mathrm{dom}(EM(I))=\overline{D/_\sim}$ where

$$t_1(x_0,\dots,x_{m-1})\sim t_2(y_0,\dots y_{n-1})$$

if

$$(t_1(a_{i_0},\dots,a_{i_{m-1}}))^{\mathscr{A}^*}=(t_2(a_{j_0},\dots,a_{j_{n-1}}))^{\mathscr{A}^*}$$

where $\langle i_0,\dots,i_{m-1},j_0,\dots,j_{n-1}\rangle$ has the same order as $\langle x_0,\dots,x_{m-1},y_0,\dots,y_{n-1}\rangle$ and $a_{i_0},\dots,a_{i_{m-1}},a_{j_0},\dots,a_{j_{n-1}}\in I_k$ for some $k\geq m+n$. By indiscernibility, the exact choice of elements does not matter, and the $*$-type of the elements determine the τ-structure of $D/_\sim$.

We show by induction on $\mathrm{card}(I)$ that $EM(I)\in\mathbb{K}$. For J a suborder of I we denote

$$A_J=\{t(y_0,\dots,y_{m-1}):t\text{ a }\tau^*\text{-term and }y_0,\dots,y_{m-1}\in J\}/_\sim.$$

We then note that for each finite sequence $J=\langle x_0,\dots,x_{n-1}\rangle$ and for every correspondingly ordered $a\in(I_{n'})^n$ with $n'\geq n$, A_J is isomorphic to

$$A(a)=\{(t(a))^{\mathscr{A}^*}:t\text{ a }\tau^*\text{-term}\}$$

which is contained in $\mathscr{A}^*$ and closed under the functions $(F_n^k)^{\mathscr{A}^*}$, $k,n<\omega$. So by 5.1 $\overline{A(a)}\restriction\tau\preccurlyeq\mathscr{A}$ and hence we can use $\overline{A(a)}$ to define the τ^*-structure on $\overline{A_J}$ such that

$\overline{A_J} \restriction \tau \in \mathbb{K}$. If for $J \subseteq J'$ we identify equivalence classes $t_1 \in A_J$ and $t_2 \in A_{J'}$ whenever $t_1 \subseteq t_2$, we get

$$A_J \subseteq A_{J'} \text{ whenever } J \subseteq J'.$$

By the universality of $\mathfrak{M}$ we may assume $\overline{A_J} \restriction \tau \preccurlyeq \mathfrak{M}$ and hence for all finite subsequences $J \subset J'$ of I we have

$$\overline{A_J} \restriction \tau \preccurlyeq \overline{A_{J'}} \restriction \tau.$$

Next let $\mathrm{card}(I) = \kappa \geq \aleph_0$ and use as induction hypothesis the property that for all suborders $J, J' \subset I$ of smaller cardinality

$$\overline{A_J} \restriction \tau \in \mathbb{K}$$

and if $J \subseteq J'$

$$\overline{A_J} \restriction \tau \preccurlyeq \overline{A_{J'}} \restriction \tau.$$

Write I as an increasing union of suborders J_i, $i < \kappa$ of smaller cardinality and note that

$$A_I = D/_\sim = \bigcup_{i<\kappa} A_{J_i}.$$

So $EM(I)$ is defined as the closure of the union of an increasing chain:

$$EM(I) = \overline{\bigcup_{i<\kappa} A_{J_i}} = \overline{\bigcup_{i<\kappa} \overline{A_{J_i}} \restriction \tau}.$$

Now if I is well-ordered and A is a countable subset of $EM(I)$, choose a subset $J \subset I$ such that each element of A is a limit of elements of the form $t(b_0, \ldots, b_{n-1})$, where t is a τ^*-term and $b_0, \ldots, b_{n-1} \in J$. J can be chosen so that $\mathrm{card}(J) \leq \mathrm{card}(A) + \aleph_0 = \aleph_0$. Let t be a τ^*-term and c a tuple from I. The Galois-type of $t(c)$ over A is determined already by its Galois-type over the set of elements $t(b_0, \ldots, b_{n-1})$. Further this type is determined by the $*$-type of $t(c)$ over these elements and hence by the $*$-type of $t(c)$ over J. By indiscernibility, the $*$-type of $t(c)$ is determined by the positions of the elements of c relative to the elements of J, and since J is well-ordered there are only $\mathrm{card}(J) + \aleph_0$ different positions. Hence there are at most $\mathrm{card}(J) + \tau = \aleph_0$ different $*$-types over J among the terms $t(c)$, and hence at most $\aleph_0$ Galois-types of elements of the form $t(c)$ over A. So the set of realized Galois-types over A is separable. □

Corollary 5.8. *If $\mathbb{K}$ is κ-categorical for some uncountable κ then $\mathbb{K}$ is ω-$\mathbf{d}$-stable.*

Proof. Assume towards a contradiction that $\mathbb{K}$ is not ω-$\mathbf{d}$-stable. Then there is $\mathscr{A} \in \mathbb{K}$ and a separable set $A \subset \mathscr{A}$ such that $\mathscr{A}$ realizes a nonseparable set of types over A. Without loss of generality $|\mathscr{A}| = \aleph_1$, so $\mathscr{A}$ can be extended to a model $\mathscr{M}$ of size κ.

On the other hand we can build an Ehrenfeucht-Mostowski model $EM(\kappa)$ of density κ. By categoricity $EM(\kappa) \cong \mathscr{M}$ but $EM(\kappa)$ realizes only a separable set of types over any countable parameter set whereas $\mathscr{M}$ realizes a non-separable set of types over $A \subset \mathscr{M}$, a contradiction. □

Note that by Theorem 4.13 this implies that if $\mathbb{K}$ is κ-categorical in some uncountable κ, then it is stable in every $\lambda = \lambda^{\aleph_0}$. Especially $\mathbb{K}$ is $\mathfrak{c}$-stable.

6 Homogeneous model theory

In this section we introduce predicates R_p for the types in order to get a first order setting and be able to use results of homogeneous model theory from [HS00].

Definition 6.1. We denote by τ^h the extension of the signature τ where we have added a predicate R_p for each Galois-type p over the empty set. We expand $\mathfrak{M}$ to $\mathfrak{M}^h$ by interpreting the predicates R_p in the obvious way, i.e. for each $a \in \mathfrak{M}$

$$\mathfrak{M}^h \models R_p(a) \text{ if and only if } \mathrm{t}^g(a/\emptyset) = p.$$

Note that by homogeneity, two elements have the same Galois-type over a set A if and only if they have the same quantifier-free τ^h-type over A. It follows that two elements have the same Galois-type over a given set if and only if they have the same $FO(\tau^h)$-type over that set and that $\mathfrak{M}^h$ is homogeneous.

We wish to use results in homogeneous model theory from [HS00]. The setting there is a stable homogeneous monster model together with its (small enough) elementary submodels. So we will assume that $\mathbb{K}$ is ω-$\mathbf{d}$-stable and hence by Theorem 4.13 $\mathfrak{c}$-stable and show that the $F^{\mathfrak{M}}_{\omega}$-saturated $\preccurlyeq$-submodels of $\mathfrak{M}$ and the $\preccurlyeq_{\omega\omega}$-submodels of $\mathfrak{M}^h$ are the same.

Lemma 6.2. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then for every metrically complete A,*

$$A \preccurlyeq_{\omega\omega} \mathfrak{M}^h \textit{ if and only if } A \restriction \tau \preccurlyeq \mathfrak{M} \textit{ and } A \textit{ is } F^{\mathfrak{M}}_{\omega}\textit{-saturated.}$$

Proof. To see that the $F^{\mathfrak{M}}_{\omega}$-saturated $\preccurlyeq$-submodels of $\mathfrak{M}$ are elementary submodels of $\mathfrak{M}^h$, let $\mathscr{A} \preccurlyeq \mathfrak{M}$, $a \in \mathscr{A}$ and $b \in \mathfrak{M}^h$. Then by $F^{\mathfrak{M}}_{\omega}$-saturation there is $b' \in \mathscr{A}$ such that $\mathrm{t}^g(ab'/\emptyset) = \mathrm{t}^g(ab/\emptyset)$ which takes care of the Tarski-Vaught criterion.

For the other direction note that for any $a \in A$, $b \in \mathfrak{M}^h$ and $R_p \in \tau^h$, there is $b' \in A$ such that

$$\mathfrak{M}^h \models R_p(ab) \text{ if and only if } \mathfrak{M}^h \models R_p(ab').$$

But this means that b and b' have the same Galois-type over a. So A is $F^{\mathfrak{M}}_{\omega}$-saturated and hence by Corollary 3.11, $A \preccurlyeq \mathfrak{M}$. □

Hence, as long as we assume a stable monster and consider only $F^{\mathfrak{M}}_{\omega}$-saturated models, the results from [HS00] apply. This way we can use the independence calculus tools developed in [HS00] and hence know that over strongly $F^{\mathfrak{M}}_{\kappa(\mathbb{K})}$-saturated models transitivity, stationarity, finite character and symmetry for $\downarrow$ hold.

Assuming ω-$\mathbf{d}$-stability we also see that the independence notions $\downarrow$ and $\downarrow^0$ coincide:

Lemma 6.3. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable. If $\mathscr{A}$ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated then it is strongly $F^{\mathfrak{M}}_{\aleph_1}$-saturated.*

Proof. Fix a and $A \subset \mathscr{A}$ such that $|A| < \aleph_1$. We wish to realize the strong type of a over A in $\mathscr{A}$. By Theorem 4.6 pick a separable $A_0 \subset \mathscr{A}$ such that $A_0 \supseteq A$ and $\mathrm{t}^g(a/\mathscr{A})$ does not split over A_0. By induction on $i < \omega$ and the $F^{\mathfrak{M}}_{\aleph_1}$-saturation of $\mathscr{A}$ let $a_i \in \mathscr{A}$ realize $\mathrm{t}^g(a/A_0 \cup \bigcup\{a_j : j < i\})$. Then $(a_i)_{i<\omega}{}^\frown\langle a\rangle$ is indiscernible over A_0, so $\mathrm{Lstp}(a_i/A_0) = \mathrm{Lstp}(a/A_0)$ and hence $\mathrm{Lstp}(a_i/A) = \mathrm{Lstp}(a/A)$. □

Theorem 6.4. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable, $\mathscr{A} \subseteq B$ and $\mathscr{A}$ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated, then for any a, $a \downarrow^0_{\mathscr{A}} B$ if and only if $a \downarrow_{\mathscr{A}} B$.*

Proof. First assume $a \downarrow_{\mathscr{A}} B$. By lemma 4.11 $a \downarrow^0_{\mathscr{A}} \mathscr{A}$. Let $\mathscr{B} \supset B$ be $F^{\mathfrak{M}}_{\aleph_1}$saturated and use theorem 4.15 to find some b realizing $\mathrm{t}^g(a/\mathscr{A})$ and satisfying $b \downarrow^0_{\mathscr{A}} \mathscr{B}$. Then since $\mathrm{t}^g(b/\mathscr{B})$ does not split strongly over $\mathscr{A}$, by [HS00, Lemma 3.2(iii)] $b \downarrow_{\mathscr{A}} \mathscr{B}$ and by monotonicity $b \downarrow_{\mathscr{A}} B$. Then by stationarity ([HS00, Lemma 3.4]) $\mathrm{t}^g(b/B) = \mathrm{t}^g(a/B)$ which proves $a \downarrow^0_{\mathscr{A}} B$.

For the other direction assume $a \downarrow^0_{\mathscr{A}} B$. Again by [HS00, Lemma 3.2(iii)], $a \downarrow_{\mathscr{A}} \mathscr{A}$. Since $\downarrow$ has built in extensions this implies the existence of some b realizing $\mathrm{t}^g(a/\mathscr{A})$ and satisfying $b \downarrow_{\mathscr{A}} B$. Then by the previous direction $b \downarrow^0_{\mathscr{A}} B$ so by stationarity (lemma 4.12) we are done. □

7 Prime models

Definition 7.1. A type $\mathrm{t}^g(a/A)$ is $F^{\mathfrak{M}}_{\kappa}$-*isolated* if there is $B \subseteq A$ of cardinality $< \kappa$ such that for all b, $\mathrm{t}^g(b/B) = \mathrm{t}^g(a/B)$ implies $\mathrm{t}^g(b/A) = \mathrm{t}^g(a/A)$. In this case we say that $\mathrm{t}^g(a/B)$ $F^{\mathfrak{M}}_{\kappa}$-isolates $\mathrm{t}^g(a/A)$.

Definition 7.2. (i) An $F^{\mathfrak{M}}_{\kappa}$-*construction* is a triple $\langle A, (a_i)_{i<\alpha}, (B_i)_{i<\alpha}\rangle$ such that $B_i \subset A \cup \bigcup\{a_j : j < i\}$ and $\mathrm{t}^g(a_i/B_i)$ $F^{\mathfrak{M}}_{\kappa}$-isolates $\mathrm{t}^g(a_i/A \cup \bigcup\{a_j : j < i\})$.

(ii) A set C is called $F^{\mathfrak{M}}_{\kappa}$-*constructible over* A if there exists an $F^{\mathfrak{M}}_{\kappa}$-construction $\langle A, (a_i)_{i<\alpha}, (B_i)_{i<\alpha}\rangle$ such that $C = A \cup \bigcup\{a_i : i < \alpha\}$.

(iii) We say that C is $F^{\mathfrak{M}}_{\kappa}$-*primary over* A if it is $F^{\mathfrak{M}}_{\kappa}$-constructible over A and $F^{\mathfrak{M}}_{\kappa}$-saturated.

(iv) A model $\mathscr{C}$ is $F^{\mathfrak{M}}_{\kappa}$-*prime over* A if it is $F^{\mathfrak{M}}_{\kappa}$-saturated, $A \subseteq \mathscr{C}$ and for every $F^{\mathfrak{M}}_{\kappa}$-saturated $\mathscr{C}' \supseteq A$ there is a $\mathbb{K}$-embedding $f : \mathscr{C} \to \mathscr{C}'$ such that $f \restriction A = \mathrm{id}$.

(v) A set C is $F^{\mathfrak{M}}_{\kappa}$-*atomic over* A if $A \subseteq C$ and for every $a \in C$, $\mathrm{t}^g(a/A)$ is $F^{\mathfrak{M}}_{\kappa}$-isolated.

Once again, we will define a metric version of a standard notion, this time of isolation.

Definition 7.3. For $A \subseteq B$ and a, we say that $\mathrm{t}^g(a/A)$ ε-*isolates* $\mathrm{t}^g(a/B)$ if for all C such that $A \subseteq C \subseteq B$ and $C \backslash A$ is finite and for all b, $\mathrm{t}^g(a/A) = \mathrm{t}^g(b/A)$ implies $\mathbf{d}(\mathrm{t}^g(a/C), \mathrm{t}^g(b/C)) < \varepsilon$.

Lemma 7.4. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable then for all $\varepsilon > 0$, a, C and finite $A \subseteq C$, there are B and b such that $A \subseteq B \subseteq C$, $B \backslash A$ is finite, $\mathrm{t}^g(b/A) = \mathrm{t}^g(a/A)$ and $\mathrm{t}^g(b/B)$ ε-isolates $\mathrm{t}^g(b/C)$.*

Proof. Assume towards a contradiction that this does not hold. Define A_n for $n < \omega$ and a_η for $\eta \in {}^{<\omega}2$ as follows:

(i) $A_0 = A$, $A_n \subseteq A_{n+1}$, $A_{n+1} \backslash A_n$ is finite,

(ii) $a_\emptyset = a$ and if length$(\eta) \geq n$ then $\mathrm{t}^g(a_\eta/A_n) = \mathrm{t}^g(a_{\eta\restriction n}/A_n)$,

(iii) $\mathbf{d}(\mathrm{t}^g(a_{\eta^\frown\langle 0\rangle}/A_{\mathrm{length}(\eta)+1}), \mathrm{t}^g(a_{\eta^\frown\langle 1\rangle}/A_{\mathrm{length}(\eta)+1})) \geq \varepsilon$.

If A_n, a_η have been defined, $\eta \in {}^n 2$, then since $\mathrm{t}^g(a_\eta/A_n)$ by our assumption does not ε-isolate $\mathrm{t}^g(a_\eta/C)$ there are a'_η and a finite A'_η such that $\mathrm{t}^g(a'_\eta/A_n) = \mathrm{t}^g(a_\eta/A_n)$ but $\mathbf{d}(\mathrm{t}^g(a'_\eta/A_n \cup A'_\eta), \mathrm{t}^g(a_\eta/A_n \cup A'_\eta)) \geq \varepsilon$. Let $a_{\eta^\frown\langle 0\rangle} = a_\eta$, $a_{\eta^\frown\langle 1\rangle} = a'_\eta$ and once a_η has been defined for all $\eta \in {}^{n+1}2$ let $A_{n+1} = A_n \cup \bigcup\{A'_\eta : \eta \in {}^n 2\}$. Since ${}^n 2$ is finite and each A'_η is finite, $A_{n+1} \backslash A_n$ is finite.

Next note that for each $\eta \in {}^\omega 2$ and each $n < \omega$, $\mathrm{t}^g(a_{\eta\restriction n}/A_n) = \mathrm{t}^g(a_{\eta\restriction n+1}/A_n)$ so by Lemma 4.14 for each $\eta \in {}^\omega 2$, there is a_η such that $\mathrm{t}^g(a_\eta/A_n) = \mathrm{t}^g(a_{\eta\restriction n}/A_n)$. Let $A^* = \bigcup_{n<\omega} A_n$. Then A^* is at most countable but for $\eta \neq \nu \in {}^\omega 2$ when $n = \min\{n : \eta(n) \neq \nu(n)\}$

$$\begin{aligned}\mathbf{d}(\mathrm{t}^g(a_\eta/A^*), \mathrm{t}^g(a_\nu/A^*)) &\geq \mathbf{d}(\mathrm{t}^g(a_\eta/A_{n+1}), \mathrm{t}^g(a_\nu/A_{n+1}))\\ &= \mathbf{d}(\mathrm{t}^g(a_{\eta\restriction n+1}/A_{n+1}), \mathrm{t}^g(a_{\nu\restriction n+1}/A_{n+1}))\\ &= \mathbf{d}(\mathrm{t}^g(a_{\eta\restriction n^\frown\langle 0\rangle}/A_{n+1}), \mathrm{t}^g(a_{\eta\restriction n^\frown\langle 1\rangle}/A_{n+1}))\\ &\geq \varepsilon.\end{aligned}$$

This contradicts ω-$\mathbf{d}$-stability. ∎

Lemma 7.5. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable and A is countable. Then for all a and $B \supseteq A$ there is b such that $\mathrm{t}^g(b/A) = \mathrm{t}^g(a/A)$ and $\mathrm{t}^g(b/B)$ is $F^{\mathfrak{M}}_{\aleph_1}$-isolated.*

Proof. Define $A_n \subseteq B$ and a_n for $n < \omega$ such that

(i) $A_0 = A$, $a_0 = a$, $A_n \subseteq A_{n+1}$ and A_n is countable,

(ii) $\mathrm{t}^g(a_{n+1}/A_n) = \mathrm{t}^g(a_n/A_n)$,

(iii) for $n > 0$, $\mathrm{t}^g(a_n/A_n)$ $\frac{1}{n}$-isolates $\mathrm{t}^g(a_n/B)$.

This is possible by the previous lemma. Let $A^* = \bigcup_{n<\omega} A_n$ and by Lemma 4.14 let b be such that $\mathrm{t}^g(b/A_n) = \mathrm{t}^g(a_n/A_n)$ for all $n < \omega$. Then A^* is countable, $\mathrm{t}^g(b/A) = \mathrm{t}^g(a/A)$ and $\mathrm{t}^g(b/A^*)$ $F^{\mathfrak{M}}_{\aleph_1}$-isolates $\mathrm{t}^g(b/B)$: If not let c be such that $\mathrm{t}^g(c/A^*) = \mathrm{t}^g(b/A^*)$ but $\mathrm{t}^g(c/B) \neq \mathrm{t}^g(b/B)$. Then there is a finite $A' \subseteq B$ such that $\mathrm{t}^g(c/A') \neq \mathrm{t}^g(b/A')$ and hence $\mathbf{d}(\mathrm{t}^g(b/A'), \mathrm{t}^g(c/A')) > 0$, contradicting $\frac{1}{n}$-isolation for n large enough. ∎

Using the previous lemma we can proceed essentially as in [She90, Theorem IV.3.1] and obtain:

Fact 7.6. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable. Then for all A there is an $F^{\mathfrak{M}}_{\aleph_1}$-primary model over A.*

Further by considering type-preserving mappings instead of elementary mappings in [She90, Theorem IV.3.10] we obtain:

Fact 7.7. *If $\mathscr{A}$ is $F^{\mathfrak{M}}_\kappa$-constructible over A and $\mathscr{B} \supset A$ is a model realizing all $F^{\mathfrak{M}}_\kappa$-isolated types over subsets of cardinality $< \kappa$, then there is a $\mathbb{K}$-embedding $f : \mathscr{A} \to \mathscr{B}$ such that $f \restriction A = \mathrm{id}$.*

Corollary 7.8. *If $\mathscr{A}$ is $F^{\mathfrak{M}}_\kappa$-primary over A then it is $F^{\mathfrak{M}}_\kappa$-prime over A.*

By a proof similar to that of [She90, Theorem IV.3.2] we obtain:

Fact 7.9. *If $\mathscr{A}$ is $F^{\mathfrak{M}}_{\aleph_1}$-primary over A then $\mathscr{A}$ is $F^{\mathfrak{M}}_{\aleph_1}$-atomic over A.*

Definition 7.10. We say that A *dominates* B *over* C, denoted $A \triangleright_C B$, if for all a, $a \downarrow_C A$ implies $a \downarrow_C B$.

Lemma 7.11. *If* $\mathbb{K}$ *is* ω-$\mathbf{d}$*-stable,* $\mathscr{A}$ *is* $F^{\mathfrak{M}}_{\aleph_1}$*-saturated and* $\mathscr{B}$ *is* $F^{\mathfrak{M}}_{\aleph_1}$*-primary over* $\mathscr{A} \cup B$*, then* $B \triangleright_{\mathscr{A}} \mathscr{B}$*.*

Proof. Assume this is not the case. Then there is some a such that $a \downarrow_{\mathscr{A}} B$ but $a \not\downarrow_{\mathscr{A}} \mathscr{B}$. Now by finite character of $\downarrow$ ([HS00, Corollary 3.5]) there is a finite $b \in \mathscr{B}$ such that $a \not\downarrow_{\mathscr{A}} b$.

Without loss of generality, we may assume B is countable, since we can restrict ourself to the part of B needed to isolate $\mathrm{t}^g(b/\mathscr{A} \cup B)$. Now choose a countable $A \subseteq \mathscr{A}$ such that

(i) $\mathrm{t}^g(b/A \cup B)$ $F^{\mathfrak{M}}_{\aleph_1}$-isolates $\mathrm{t}^g(b/\mathscr{A} \cup B)$,

(ii) for all $d \in B$, $\mathrm{t}^g(d/\mathscr{A} \cup a)$ does not split strongly over A,

(iii) $\mathrm{t}^g(b/\mathscr{A})$ does not split over A.

(i) can be achieved since $b \in \mathscr{B}$, (ii) follows from the assumption that $a \downarrow_{\mathscr{A}} \mathscr{B}$ by symmetry and finite character and (iii) we get from Theorem 4.6.

Now by 6.3 $\mathscr{A}$ is strongly $F^{\mathfrak{M}}_{\aleph_1}$-saturated and we may choose $a' \in \mathscr{A}$ with $\mathrm{Lstp}(a'/A) = \mathrm{Lstp}(a/A)$. By (ii) $\mathrm{t}^g(a'/A \cup d) = \mathrm{t}^g(a/A \cup d)$ for all $d \in B$ so by homogeneity $\mathrm{t}^g(a'/A \cup B) = \mathrm{t}^g(a/A \cup B)$. Fix $f \in \mathrm{Aut}(\mathfrak{M}/A \cup B)$ mapping a to a'. By (iii) and [HS00, Lemma 3.2(iii)], $b \downarrow_A \mathscr{A}$ so $b \downarrow_A a'$. However, $b \not\downarrow_{\mathscr{A}} a$ so especially $b \not\downarrow_A a$ and $f(b) \not\downarrow_A a'$, contradicting (i). □

8 Categoricity transfer

Remark 8.1. If $\mathbb{K}$ is κ-categorical for some $\kappa = \kappa^{\aleph_0}$ then by 5.8 $\mathbb{K}$ is ω-$\mathbf{d}$-stable and by 4.13 it is stable in every $\lambda = \lambda^{\aleph_0}$. Hence it is κ-stable. Since the density and cardinality of κ-sized models is the same, we may use [HS00, Theorem 3.14] to find a $F^{\mathfrak{M}}_{\kappa}$-saturated model of size κ so the unique model of size κ is saturated and hence every model of size $\geq \kappa$ is $F^{\mathfrak{M}}_{\kappa}$-saturated.

Lemma 8.2. *If* $\mathbb{K}$ *is* κ*-categorical for some* $\kappa^{\aleph_0} = \kappa$ *then there is some* $\xi < \beth_{\mathfrak{c}^+}$ *such that every model of cardinality at least* ξ *is* $F^{\mathfrak{M}}_{\aleph_1}$*-saturated.*

Proof. Assume this is not the case. Then if $(\lambda_i)_{i<\mathfrak{c}^+}$ is an increasing cardinal sequence such that $\bigcup_{i<\mathfrak{c}^+} \lambda_i = \beth_{\mathfrak{c}^+}$ and $\lambda_i < \beth_{\mathfrak{c}^+}$, we can for each $i < \mathfrak{c}^+$ find a model $\mathscr{A}_i$ such that $\mathscr{A}_i$ has cardinality λ_i and is not $F^{\mathfrak{M}}_{\aleph_1}$-saturated. By Theorem 3.7 $\mathscr{A}_i$ is not $F^{\mathfrak{M}}_{\aleph_1}$-$\mathbf{d}$-saturated so there is a countable $A_i \subset \mathscr{A}_i$, a type $p_i \in S(A_i)$ and some $\varepsilon_i > 0$ such that all types $q \in S(A_i)$ realized in $\mathscr{A}_i$ satisfy $\mathbf{d}(p_i, q) \geq \varepsilon_i$. We will refer to this property as omitting p_i at distance ε_i.

By the categoricity assumption $\mathbb{K}$ is ω-$\mathbf{d}$-stable and hence also $\mathfrak{c}$-stable. Hence there are at most $\mathfrak{c}$ types over the empty set and since by homogeneity the type of a countable set is determined by the types of its finite subsets, there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ countable types over the empty set. Hence $\mathfrak{c}^+$ many sets A_i have the same type over the empty set and we can denote one of these sets by A and find automorphisms f_i of the monster model mapping the other sets A_i of same type onto A. Then the corresponding models $\mathscr{A}_i$ are mapped to models $f_i(\mathscr{A}_i)$ containing A and omitting some type over A at distance ε_i. Again we use $\mathfrak{c}$-stability to see that there are at most $\mathfrak{c}$ different types over A and hence $\mathfrak{c}^+$ many models omit the same type p and finally by the pigeonhole principle $\mathfrak{c}^+$ many models omit p at the same fixed distance ε. Among these models rename and enumerate $\mathfrak{c}^+$ many as $(\mathscr{A}_i)_{i<\mathfrak{c}^+}$, such that

(i) $A \subseteq \mathscr{A}_i$ and $\mathscr{A}_i$ omits p at distance ε,

(ii) $|\mathscr{A}_i| \geq \beth_{\omega \cdot i}$.

For each $\mathscr{A}_i$, add constant symbols for the elements of A and Skolem functions to obtain expansions $\mathscr{A}_i^*$ as in 5.1. Then by a modelwise version of Fact 5.6, we obtain sequences I_n, each inside some $\mathscr{A}_i$, which are $*$-n-order-indiscernible over A and have the property that for $n > m$, m-tuples of I_n and I_m have the same $*$-type over A in their corresponding models.

Now we can build an Ehrenfeucht-Mostowski model $EM(\kappa)$ as in Theorem 5.7 consisting of equivalence classes of τ^*-terms (and elements obtained in taking the completion) with for τ^*-terms t_1 and t_2

$$t_1(x_0, \ldots, x_{n-1}) = t_2(y_0, \ldots, y_{m-1})$$

if

$$(t_1(a_{i_0}, \ldots, a_{i_{n-1}}))^{\mathscr{A}_i^*} = (t_2(b_{i_0}, \ldots, b_{i_{m-1}}))^{\mathscr{A}_i^*}$$

where $a_i, b_i \in I_k \subset \mathscr{A}_i$ for $k \geq n+m$. Since the indiscernible sequences agree on short enough tuples, the equivalence relation is well-defined. Also if a' and a are tuples from different indiscernible sequences with $\text{length}(a') < \text{length}(a)$ then the model $\overline{A(a')}$ defined in the proof of 5.7 embeds into $\overline{A(a)}$. Hence the construction of $EM(\kappa)$ can be carried out as in 5.7 and we obtain a model of size κ which because of the added constants contains A. Further since every element in $\text{SH}(\kappa)$ is the interpretation of an τ^*-term, interpreted as in some $\mathscr{A}_i$, all these elements realize a type at distance $\geq \varepsilon$ from p. Hence $EM(\kappa)$ omits p. But this contradicts the categoricity assumption since by 8.1 the unique model of size κ must be saturated. □

For our forthcoming constructions we need the classical notion of a Morley sequence. In our settings it means the following:

Definition 8.3. Let $A \subseteq B$ and let $p \in S(B)$. We call a sequence $(a_i)_{i<\alpha}$ a *Morley sequence for p over A* if $\text{t}^g(a_i/B) = p$ for all $i < \alpha$ and $a_i \downarrow_A B \cup \bigcup_{j<i} a_j$ for all $i < \alpha$. The Morley sequence is *nontrivial* if $a_i \notin A$ for all $i < \alpha$.

Remark 8.4. Note that without further requirements for the set A Morley sequences over A do not necessarily exist. However if $\mathbb{K}$ is ω-$\mathbf{d}$-stable and A is an $F^{\mathfrak{M}}_{\aleph_1}$-saturated model we can construct a Morley sequence over A since then $a_0 \downarrow_A A$ for any a_0 and we can choose a_α such that $\text{t}^g(a_\alpha/A \cup \{a_i : i < \alpha\}) = \text{t}^g(a_0/A \cup \{a_i : i < \alpha\}$ and $\text{t}^g(a_\alpha/D)$ does not split strongly over $C \subset A$ (see definition 4.9) for some $F^{\mathfrak{M}}_{\aleph_1}$saturated D containing $A \cup \{a_i : i < \alpha\}$.

Furthermore by stationarity, if A is $F^{\mathfrak{M}}_{\aleph_1}$-saturated then for a given type $p \in S(A)$ the Morley sequence over A is unique up to A-isomorphism, i.e. if I and J are Morley sequences for p over A with $\text{length}(I) = \text{length}(J)$ then there is $f \in \text{Aut}(\mathfrak{M}/A)$ mapping I to J.

Theorem 8.5 (Unidimensionality). *Assume $\mathbb{K}$ is κ-categorical for some $\kappa^{\aleph_0} = \kappa > \aleph_1$. For all $F^{\mathfrak{M}}_{\aleph_1}$-saturated $\mathscr{A}$ and unbounded $p, q \in S(\mathscr{A})$, if $(a_i)_{i<\omega}$ is a nontrivial Morley sequence for p then q has a realization in any $F^{\mathfrak{M}}_{\aleph_1}$-primary model over $\mathscr{A} \cup \{a_i : i < \omega\}$.*

Proof. Let $\mathscr{A}$ be $F^{\mathfrak{M}}_{\aleph_1}$-saturated and let p and q be unbounded types over $\mathscr{A}$. By 8.1 $\mathbb{K}$ is ω-$\mathbf{d}$-stable and $\mathfrak{c}$-stable so by 4.6 we can find a countable $A \subset \mathscr{A}$ such that p and q do not split over A. By ω-$\mathbf{d}$-stability we can extend A by an $\aleph_1$-length construction to an $F^{\mathfrak{M}}_{\aleph_1}$-saturated model $\mathscr{A}'$ of density $\aleph_1$. By 4.8 and 6.3, $\mathscr{A}'$ is strongly $F^{\mathfrak{M}}_{\kappa(\mathbb{K})}$-saturated so by [HS00, 3.2] we can construct a nontrivial Morley sequence $I = (b_i)_{i<\kappa}$ for p over $\mathscr{A}'$. By Theorem 7.6 let $\mathscr{B}$ be an $F^{\mathfrak{M}}_{\aleph_1}$-primary model over $\mathscr{A}' \cup I$. Then $|\mathscr{B}| \geq \kappa$ so by Remark 8.1 $\mathscr{B}$ is $F^{\mathfrak{M}}_{\aleph_2}$-saturated. So $\mathscr{B}$ realizes $q \restriction \mathscr{A}'$ by, say, a. Now by the construction $I \downarrow_{\mathscr{A}'} \mathscr{A}$, so by symmetry ([HS00, Corollary 3.5 and Lemma 3.6]) $\mathscr{A} \downarrow_{\mathscr{A}'} I$. Further by 7.11 $I \rhd_{\mathscr{A}'} \mathscr{B}$, so $\mathscr{A} \downarrow_{\mathscr{A}'} \mathscr{B}$ and hence $\mathscr{B} \downarrow_{\mathscr{A}'} \mathscr{A}$ and finally $a \downarrow_{\mathscr{A}'} \mathscr{A}$.

To see that $\mathrm{t}^g(a/\mathscr{A}) = q$, let $b \in \mathfrak{M}$ realize q. By [HS00, Lemma 3.4] types over strongly $F^{\mathfrak{M}}_{\aleph_1}$-saturated models are stationary and by the choice of $\mathscr{A}'$, $b \downarrow_{\mathscr{A}'} \mathscr{A}$, so $\mathrm{t}^g(a/\mathscr{A}) = \mathrm{t}^g(b/\mathscr{A}) = q$. We finally want to find a realization of q already in a primary model over a countable Morley sequence. By 7.9, $\mathscr{B}$ is $F^{\mathfrak{M}}_{\aleph_1}$-atomic over $\mathscr{A}' \cup I$ so there are countable $A' \subset \mathscr{A}'$ and $J \subset I$ such that $\mathrm{t}^g(a/A' \cup J)$ isolates $\mathrm{t}^g(a/\mathscr{A}' \cup I)$. Hence if $\mathscr{B}'$ is $F^{\mathfrak{M}}_{\aleph_1}$-primary over $\mathscr{A}' \cup J$ it realizes $\mathrm{t}^g(a/\mathscr{A}' \cup J)$ by, say, a' and as above by symmetry and dominance $a' \downarrow_{\mathscr{A}'} \mathscr{A}$ and by stationarity $\mathrm{t}^g(a'/\mathscr{A}) = q$.

We claim J has the required properties. So let $\mathscr{C}$ be $F^{\mathfrak{M}}_{\aleph_1}$-primary over $\mathscr{A} \cup J$. Then since $\mathscr{B}'$ by 7.8 is $F^{\mathfrak{M}}_{\aleph_1}$-prime over $\mathscr{A}' \cup J \subset \mathscr{A} \cup J$, there is a $\mathbb{K}$-embedding $f : \mathscr{B}' \to \mathscr{C}$ such that $f \restriction \mathscr{A}' \cup J = \mathrm{id}$. Then $f(\mathscr{B}')$ is still $F^{\mathfrak{M}}_{\aleph_1}$-primary over $\mathscr{A}' \cup J$ so as above $f(a') \downarrow_{\mathscr{A}'} \mathscr{A}$ and $\mathrm{t}^g(f(a')/\mathscr{A})$ must be q.

Finally if $J' = (a_i)_{i<\omega}$ is any nontrivial Morley sequence for p then since $\mathrm{t}^g(a_i/\mathscr{A})$ does not split over A, by transitivity ([HS00, Lemma 3.8(ii)]) J' is Morley over $\mathscr{A}'$ and by the uniqueness of Morley sequences we see that the claim holds for J'. □

We are now ready to glue together the pieces into our main theorem.

Theorem 8.6. *Assume $\mathbb{K}$ is κ-categorical for some $\kappa = \kappa^{\aleph_0} > \aleph_1$. Then there is $\xi < \beth_{\mathfrak{c}^+}$ such that $\mathbb{K}$ is categorical in all λ satisfying*

(i) $\lambda \geq \min\{\xi, \kappa\}$,

(ii) $\lambda^{\aleph_0} = \lambda$,

(iii) *for all* $\zeta < \lambda$, $\zeta^{\aleph_0} < \lambda$.

Proof. Assume λ is as in the claim. We prove that all models of density λ are saturated, from which the claim follows. So let $|\mathscr{A}| = \lambda$, $B \subset \mathscr{A}$, $|B| < \lambda$ and $q \in S(B)$.

By (i) and either Remark 8.1 or Lemma 8.2 $\mathscr{A}$ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated. By 8.1 $\mathbb{K}$ is ω-$\mathbf{d}$-stable and λ-stable for every $\lambda = \lambda^{\aleph_0}$, so by (iii) we may assume that B is also $F^{\mathfrak{M}}_{\aleph_1}$-saturated and by 6.3 B is in fact strongly $F^{\mathfrak{M}}_{\aleph_1}$-saturated.

First assume λ is regular. Choose $\mathscr{B}_i$ and b_i, for $i < \lambda$ such that

(i) $\mathscr{B}_0 = B$,

(ii) $b_i \in \mathscr{A} \backslash \mathscr{B}_i$,

(iii) $\mathscr{B}_i \subseteq \mathscr{A}$,

(iv) $\mathscr{B}_{i+1}$ is $F^{\mathfrak{M}}_{\aleph_1}$-primary over $\mathscr{B}_i \cup b_i$,

(v) $|\mathscr{B}_i| < \lambda$,

(vi) for limit i, $\mathscr{B}_i \supset \bigcup_{j<i} \mathscr{B}_j$ and is (strongly) $F^{\mathfrak{M}}_{\aleph_1}$-saturated and if $cf(i) \geq \aleph_1$ then $\mathscr{B}_i = \bigcup_{j<i} \mathscr{B}_j$.

By Theorem 4.6 for each $i < \lambda$ choose $A_i \subset \mathscr{B}_i$ such that $\mathrm{t}^g(b_i/\mathscr{B}_i)$ does not split over A_i. Let $f(i)$ be the least j for which $A_i \subset \mathscr{B}_j$. Then for $cf(i) \geq \aleph_1$, $f(i) < i$ and since the set of indices i with cofinality $\geq \aleph_1$ forms a stationary subset of λ, we can use Fodor's lemma to find a stationary set S and an $i^* < \lambda$ such that $A_i \subset \mathscr{B}_{i^*}$ for all $i \in S$, and we may choose i^* such that $\mathscr{B}_{i^*}$ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated. Note that S must be of power λ. Now there are only $|\mathscr{B}_{i^*}|^{\aleph_0} < \lambda$ types over $\mathscr{B}_{i^*}$, so for λ many $i \in S$ the elements b_i have the same type over $\mathscr{B}_{i^*}$. Since for these indices i, $\mathrm{t}^g(b_i/\mathscr{B}_i)$ does not split over $A_i \subset \mathscr{B}_{i^*}$, by [HS00, Lemma 3.2] we have $b_i \downarrow_{\mathscr{B}_{i^*}} \mathscr{B}_i$, i.e. $b_i \downarrow_{\mathscr{B}_{i^*}} \mathscr{B}_{i^*} \cup \{b_j : j < i\}$. Hence we have found an infinite nontrivial Morley sequence I over $\mathscr{B}_{i^*} \supset B$ and by 8.5 we are done.

For non-regular cardinals λ, saturation, and hence categoricity, is proved by induction. So assume λ is singular and satisfies the conditions (i) to (iii). We may assume $\lambda > \min\{\xi, \kappa\}$ in (i) (if $\lambda = \kappa$ we are done and if the ξ given by Lemma 8.2 is not regular we can replace it with ξ^+). Let $\mathscr{A}$ be a model with $|\mathscr{A}| = \lambda$ and $A \subset \mathscr{A}$ a subset of density $\zeta < \lambda$ with $\zeta \geq \min\{\xi, \kappa\}$. By the assumptions on λ, $\zeta^{\aleph_0} < \lambda$ and since λ must be a limit cardinal we also have $(\zeta^{\aleph_0})^+ < \lambda$. Hence by Löwenheim-Skolem we can find a model of density $(\zeta^{\aleph_0})^+$ inside $\mathscr{A}$ containing A. But by the induction hypothesis this model is saturated and we are done. □

9 Examples

This section has two objectives. On one hand we see how homogeneous metric abstract elementary classes relate to the positive bounded model theory developed by Henson and Iovino and Ben-Yaacov's compact abstract theories or cats. On the other hand we give two concrete example classes, the class of all Banach spaces and the class of all spaces isometrically isomorphic to a (possibly transfinite) ℓ^p-space for a fixed $p \neq 2$. These examples show what the additional assumptions in 2.13 mean in practice.

9.1 Positive bounded theories and cats

In [HI02] Henson and Iovino develop model theory for positive bounded theories together with the notions of approximate satisfaction and approximate elementary equivalence. For a fixed positive bounded theory T the class $\mathfrak{C}$ of models approximately satisfying T does not as such form a MAEC since the models need not be complete. However the subclass of complete models in $\mathfrak{C}$ is a homogeneous MAEC with $\mathrm{LS}^{\mathrm{d}} = \aleph_0$, where $\preccurlyeq$ is interpreted by the approximate elementary submodel relation. Unions of chains and Löwenheim-Skolem numbers with respect to densities are treated in [HI02], homogeneity follows from compactness and the perturbation property from the so called Perturbation lemma ([HI02] Proposition 5.15).

The other main approach to the model theory of metric structures is Ben-Yaacov's notion of compact abstract theories or cats. In [BY03] he defines a cat among others as a positive Robinson theory. In [BY05] he shows that every countable Hausdorff cat admits a metric. A Hausdorff cat is a cat where the type-spaces are Hausdorff in the topology induced by partial types as closed sets.

Ben-Yaacov proves an analogue of Morley's categoricity transfer theorem in [BY05]. He considers the collection of completions of so-called premodels inside a universal domain. A premodel is defined as a subset M of the universal domain such that for every $n < \omega$ the set of types over M realized in M are dense in $S^n(M)$. The existence of a language and compactness ensures that the class of complete models forms a homogeneous MAEC. The perturbation property follows from Theorem 2.20 of [BY05] where it is shown that the metric topology coincides with the logic topology on the model. This topology is defined by letting the closed sets be the sets definable by partial types.

9.2 The class of all Banach spaces

A basic example of a homogeneous metric abstract elementary class is the class of all Banach spaces with the (closed) subspace relation for $\preccurlyeq$.

Conditions (i)-(iii) and (v) of the definition of a MAEC (Definition 2.1) are trivial. Furthermore the only functions we have are the vector space operations and the norm, and hence the completion of the normed space obtained as a union of an increasing chain clearly exists, is unique, and satisfies the demands in (iv).

The Löwenheim-Skolem number of the class is $\aleph_0$ since clearly the smallest Banach space containing a given nonempty set A is

$$\overline{\mathrm{Span}(A)}.$$

Since this set is obtained as the closure of the set of rational linear combinations of elements of A, the space has density at most $|A| + \aleph_0$.

We thus see that the class is a metric abstract elementary class. We now turn our attention to the properties of a homogeneous MAEC.

Clearly there are arbitrary large Banach spaces. For the joint embedding property we define the direct sum of two Banach spaces $\mathscr{A}$ and $\mathscr{B}$ as

$$\mathscr{A} \times \mathscr{B}$$

with component-wise vector space operations and with the norm

$$\|(a, b)\| = \|a\|_{\mathscr{A}} + \|b\|_{\mathscr{B}}.$$

Now any two Banach spaces can be embedded into the direct sum of (disjoint copies of) them by the canonical embeddings

$$i : \mathscr{A} \to \mathscr{A} \times \mathscr{B},\ i(a) = (a, 0) \text{ and } j : \mathscr{B} \to \mathscr{A} \times \mathscr{B},\ j(b) = (0, b).$$

For the amalgamation property we use a construction by Kislyakov. This actually gives a stronger form of amalgamation namely one for the functional analytic notion for an isomorphism instead of an isometry.

Definition 9.1. Let f be a continuous linear bijection from $\mathscr{A}$ onto $\mathscr{B}$. Then

$$\|f\| = \sup\{\|f(x)\| : x \in \mathscr{A}, \|x\| \leq 1\} < \infty.$$

If also the inverse function f^{-1} is continuous and

$$C \geq \|f\|\,\|f^{-1}\|$$

we call f a C*-isomorphism*. We call a C-isomorphism onto a subspace of $\mathscr{B}$ a C-embedding into $\mathscr{B}$.

The functional analytic notion of an isomorphism is a C-isomorphism for some C. Note that if f is a 1-isomorphism, it is of the form cf' where c is a constant and f' is an isometry.

Lemma 9.2. *Suppose $\mathscr{A} \preccurlyeq \mathscr{B}$ (in the class of all Banach spaces) and $f : \mathscr{A} \to \mathscr{A}'$ is a C-embedding. Then there are $\mathscr{B}' \succcurlyeq \mathscr{A}'$ and $g : \mathscr{B} \to \mathscr{B}'$ such that $g \restriction \mathscr{A} = f$ and $\|g\| = \|f\|$, $\|g^{-1}\| = \|f^{-1}\|$. Especially g is a C-embedding.*

Proof. Let $\mathscr{A}$, $\mathscr{B}$, f and $\mathscr{A}'$ be as in the claim and denote $f' = \frac{f}{\|f\|}$. By the Kislyakov construction described in [CG97] Chapter 1.3, if we let $\Delta = \{\langle a, -f'(a)\rangle : a \in \mathscr{A}\}$ and define

$$\mathscr{C} = (\mathscr{B} \times \mathscr{A}')/\Delta$$

then $\mathscr{A}'$ embeds isometrically into $\mathscr{C}$ and there is an embedding $j_{\mathscr{B}}$ of $\mathscr{B}$ into $\mathscr{C}$ satisfying

$$\|j_{\mathscr{B}}\| \leq 1 \text{ and } \|j_{\mathscr{B}}^{-1}\| \leq \|(f')^{-1}\| = \|f\|\,\|f^{-1}\|.$$

We then choose an isometrically isomorphic copy $\mathscr{B}'$ of $\mathscr{C}$ and an isometric isomorphism $h : \mathscr{C} \to \mathscr{B}'$ satisfying $h \circ j_{\mathscr{A}'} = \mathrm{id}_{\mathscr{A}'}$. Defining $g = h \circ \|f\| j_{\mathscr{B}}$ gives the required embedding. □

Corollary 9.3. *The class of all Banach spaces has the amalgamation property.*

Proof. This is seen by taking f to be an isometry. Then also g is one. □

Note that since all Banach spaces have the trivial space $\{0\}$ as a subspace the joint embedding property follows from amalgamation and the construction above actually is the direct sum of $\mathscr{B}$ and $\mathscr{A}'$.

Lemma 9.4. *The class of all Banach spaces is homogeneous.*

Proof. Assume $(a_i)_{i<\alpha} \subset \mathscr{A}$ and $(b_i)_{i<\alpha} \subset \mathscr{B}$ are such that for all $n < \omega$ and $i_0, \ldots, i_{n-1}$ there are a Banach space $\mathscr{C}$ and isometric embeddings $f : \mathscr{A} \to \mathscr{C}$, $g : \mathscr{B} \to \mathscr{C}$ such that

$$f(a_{i_0}, \ldots, a_{i_{n-1}}) = g(b_{i_0}, \ldots, b_{i_{n-1}}).$$

Now for each finite $I \subset \alpha$ these embeddings define an isometric isomorphism $h_I = g^{-1} \circ f :$ $\mathrm{Span}\{a_i : i \in I\} \to \mathrm{Span}\{b_i : i \in I\}$ and for $I \neq I'$ the mappings h_I and $h_{I'}$ agree on elements in the intersection of their domains, since both map a_i to b_i for $i \in I \cap I'$. Further note that $\mathrm{Span}\{a_i : i < \alpha\} = \bigcup\{\mathrm{Span}\{a_i : i \in I\} : I \subset \alpha, |I| < \aleph_0\}$ since the set contains just finite linear combinations. Hence $h = \bigcup\{h_I : I \subset \alpha, |I| < \aleph_0\}$ is an isometric isomorphism $h : \mathrm{Span}\{a_i : i < \alpha\} \to \mathrm{Span}\{b_i : i < \alpha\}$ and can be extended to an isometric isomorphism between the closures. So

$$\mathscr{A} \succcurlyeq \overline{\mathrm{Span}\{a_i : i < \alpha\}} \cong \overline{\mathrm{Span}\{b_i : i < \alpha\}} \preccurlyeq \mathscr{B}$$

and the claim follows by amalgamation. □

Remark 9.5. The proof above actually shows that any class of Banach-spaces, which is closed under the closed subspace relation and satisfies the amalgamation and joint embedding properties, is homogeneous (when $\preccurlyeq$ is the subspace relation). Especially the class of Hilbert spaces together with the subspace relation forms a homogeneous MAEC, since amalgamation and joint embedding is trivially achieved by considering orthogonal bases.

The properties so far assure the existence of a homogeneous monster model. Our last property is the perturbation property.

Lemma 9.6. *The class of all Banach spaces has the perturbation property.*

Proof. Let $A \subset \mathfrak{M}$ and let $(a_i)_{i<\omega}$ be a convergent sequence in $\mathfrak{M}$ with $\mathrm{t}^g(a_i/A) = \mathrm{t}^g(a_j/A)$ for all $i, j < \omega$. Denote by a the limit of $(a_i)_{i<\omega}$. We wish to show that $\mathrm{Span}(A \cup a)$ is isometrically isomorphic to $\mathrm{Span}(A \cup a_0)$. For this let for each $i < \omega$, f_i be an isometric automorphism mapping a_0 to a_i and fixing A pointwise. Further let $\mathscr{A}_0 = \mathrm{Span}(A \cup a_0)$. Now note that since $(f_i(a_0))_{i<\omega}$ converges to a and $f_i \restriction A = \mathrm{id}$ for every $i < \omega$, $(f_i(b))_{i<\omega}$ converges for every $b \in \mathscr{A}_0$. More specifically, if $b = \sum_{j \leq n} r_j x_j$ for some $r_j \in \mathbb{R}$ and $x_j \in A \cup a_0$ then by linearity of f_i

$$f_i\left(\sum_{j\leq n} r_j x_j\right) = \sum_{j \leq n} r_j f_i(x_j)$$

and $f_i(x_j) = x_j$ if $x_j \in A$ and if $x_j = a_0$, $(f_i(x_j))_{i<\omega}$ converges to a.

Hence we conclude that $\mathscr{A}_0$ is isometrically isomorphic to $\mathrm{Span}(A \cup a)$, so their closures are isometrically isomorphic $\preccurlyeq$-submodels of $\mathfrak{M}$ and hence by amalgamation and homogeneity, there is an isometric automorphism F of $\mathfrak{M}$ mapping $\overline{\mathscr{A}_0}$ onto $\overline{\mathrm{Span}(A \cup a)}$, especially $F(a_0) = a$. □

Having put all the pieces together we conclude that the class of all Banach spaces with the subspace relation for $\preccurlyeq$ is a homogeneous metric abstract elementary class with LS^{d}-number $\aleph_0$.

9.3 A categorical class of Banach spaces

In this section we present an example of a categorical homogeneous MAEC consisting of Banach space structures which are not Hilbert spaces. Note that by Remark 9.5 the Hilbert spaces together with the subspace relation forms a homogeneous MAEC and this class clearly is categorical. For the example of this section we, however, need a stronger relation for $\preccurlyeq$.

We need the notion of a generalized series. In [Sin81] (Chapter III.17) this is defined as follows:

Definition 9.7. Consider the ordinals equipped with the order topology, i.e. the topology generated by the open intervals $(\beta, \gamma) = \{i : \beta < i < \gamma\}$. If $(y_i)_{i<\alpha}$ is a sequence in a Banach space E, we say that the series $\sum_{i<\alpha} y_i$ converges to an element $x \in E$ and write

$$x = \sum_{i<\alpha} y_i,$$

if there exists a continuous function $S : \alpha + 1 \to E$ such that

(i) $S(0) = 0$,

(ii) $S(\alpha) = x$,

(iii) $S(i+1) = S(i) + y_i$ for each $i < \alpha$.

In the case of $\alpha = \omega$ the definition coincides with the standard definition of a series. It also holds that for a convergent series

$$\sum_{i<\alpha} y_i = x,$$

only countably many of the terms y_i can be nonzero.

Transfinite series of reals will be used in the definition of generalized ℓ^p-spaces.

Definition 9.8. Let $1 \leq p < \infty$. For I a well-order we define the ℓ^p-space over I as the space

$$\ell^p(I) = \{(a_i)_{i\in I} : a_i \in \mathbb{R}, \sum_{i\in I} |a_i|^p < \infty\},$$

equipped with the norm

$$\|(a_i)_{i\in I}\| = \left(\sum_{i\in I} |a_i|^p\right)^{\frac{1}{p}}.$$

We will consider the class of spaces isometrically isomorphic to some $\ell^p(I)$ for a fixed p, $1 \leq p < \infty$, $p \neq 2$. Note that if $I \subset J$ there is a natural embedding $f : \ell^p(I) \to \ell^p(J)$ defined by $f((a_i)_{i\in I}) = (b_i)_{i\in J}$ where

$$b_i = \begin{cases} a_i & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we may consider $\ell^p(I)$ as a subspace of $\ell^p(J)$ and we will, with a slight abuse of notation, write $\ell^p(I) \subseteq \ell^p(J)$ for situations like this.

We start with the notion of a basis. What we call a basis is in [Sin81] referred to as a *transfinite basis*. If $\alpha = \omega$ the definition gives a Shauder basis.

Definition 9.9. Let X be a Banach space. We call a sequence $(e_i)_{i<\alpha}$ a *basis* of X if for all $x \in X$, there exists a unique sequence $(a_i)_{i<\alpha}$ of real numbers such that $x = \sum_{i<\alpha} a_i e_i$.

In classical ℓ^p-spaces there is a natural notion of a *standard basis*. The notion generalizes to $\ell^p(I)$-spaces as $(e_i)_{i\in I}$, where $e_i = (\delta_{ij})_{j\in I}$ and δ_{ij} is the Kronecker delta. Now the formal definition of our class is the following:

Definition 9.10. Let $(\mathbb{K}_{\ell^p}, \preccurlyeq)$ (where $1 \leq p < \infty$, $p \neq 2$) be the class of all structures of the form

$$\mathscr{A} = \langle E_{\mathscr{A}}, \mathbb{R}_{\mathscr{A}} \rangle,$$

where $E_{\mathscr{A}}$ is a Banach space isometrically isomorphic to some $\ell^p(I)$ and $\mathbb{R}_{\mathscr{A}}$ is isometrically isomorphic to $\mathbb{R}$, equipped with the normed space structure of $E_{\mathscr{A}}$ and the structure of the ordered field $\mathbb{R}_{\mathscr{A}}$.

As a shorthand we write $\mathscr{A} \cong \ell^p(I)$ for $E_{\mathscr{A}} \cong \ell^p(I)$. Further if $f : \ell^p(I) \to \mathscr{A}$ is an isometric isomorphism we denote for each $J \subseteq I$,

$$\hat{J}_f = \{f(e_j) : j \in J\}$$

and

$$\widetilde{\ell^p}(\hat{J}_f) = \{\sum_{i\in J} a_i f(e_i) : (a_i)_{i\in J} \in \ell^p(J)\},$$

where $(e_i)_{i\in I}$ is the standard basis of $\ell^p(I)$.

Now we can define for $\mathscr{A}, \mathscr{B} \in \mathbb{K}_{\ell^p}$,

$$\mathscr{A} \preccurlyeq \mathscr{B}$$

if whenever J is a well-order such that $\ell^p(J) \cong \mathscr{B}$ and f is an isometric isomorphism witnessing this then $\mathscr{A} = \widetilde{\ell^p}(\hat{J}_f \cap \mathscr{A})$.

The fact that this definition of $\preccurlyeq$ makes sense depends heavily on Banach's classification of the isometric automorphisms of ℓ^p-spaces for $1 \leq p < \infty$, $p \neq 2$. Note that the standard basis of an ℓ^p-space is *unconditional*, i.e. for any convergent sequence $\sum_{i<\alpha} a_i e_i$, the series $\sum_{i<\alpha} a_{\sigma(i)} e_{\sigma(i)}$ converges to the same limit for all permutations σ of α. Hence when σ is a permutation of I and $\varepsilon_i \in \{-1, 1\}$ for each $i \in I$ then all mappings U of the form

$$U(\sum_{i \in I} a_i e_i) = \sum_{i \in I} \varepsilon_i a_i e_{\sigma(i)},$$

where $a_i \in \mathbb{R}$, are isometric automorphisms of $\ell^p(I)$. By Banach's classification result, this is is the only possible form for an isometric automorphism. The theorem is proved in Chapter XI.5 of [Ban78]. We state it in a different form, but the proof is essentially the same.

Theorem 9.11. *If $1 \leq p < \infty$, $p \neq 2$ then each isometric automorphism of $\ell^p(I)$ is of the form*

$$U(\sum_{i \in I} a_i e_i) = \sum_{i \in I} \varepsilon_i a_i e_{\sigma(i)},$$

for $\sum_{i \in I} a_i e_i \in \ell^p(I)$, where $(e_i)_{i \in I}$ is the standard basis of $\ell^p(I)$, σ is a permutation of I and for each $i \in I$, $\varepsilon_i \in \{-1, 1\}$.

Remark 9.12. From the result above it follows that if $\mathscr{A}$ is a model in our class and $B \subset \mathscr{A}$ is the image of the standard basis under some isometric isomorphism $f : \ell^p(I) \to \mathscr{A}$ then B as a set is unique up to change of sign. So although we do not have a unique standard basis in $\mathscr{A}$ there is a unique set, $\mathbb{B}(\mathscr{A}) = \{\varepsilon b : b \in B, \varepsilon \in \{-1, 1\}\}$, such that if $g : \ell^p(J) \cong \mathscr{A}$ then $\hat{J}_g = g(\{e_j : j \in J\}) \subset \mathbb{B}(\mathscr{A})$ and for every $a \in \mathbb{B}(\mathscr{A})$, $a \in \hat{J}_g$ if and only if $-a \notin \hat{J}_g$. Now if $\mathscr{A}, \mathscr{B} \in \mathbb{K}_{\ell^p}$, $\mathscr{A} \preccurlyeq \mathscr{B}$ can be stated differently as $\mathbb{B}(\mathscr{A}) = \mathbb{B}(\mathscr{B}) \cap \mathscr{A}$.Note that this implies that e.g. block-base constructions generally do not give $\preccurlyeq$-submodels. This also means that the $\mathbb{K}_{\ell^p}$-embeddings $\mathscr{A} \to \mathscr{B}$ are precisely the isometric embeddings mapping $\mathbb{B}(\mathscr{A})$ into $\mathbb{B}(\mathscr{B})$.

Notation 9.13. Although for $\ell^p(I) \cong \mathscr{A}$ we do not have a unique image of the standard basis in $\mathscr{A}$ we will, for a given isometric isomorphism $f : \ell^p(I) \to \mathscr{A}$ call $\hat{I}_f = f(\{e_i : i \in I\})$ *a standard basis of* $\mathscr{A}$. Hence $\mathbb{B}(A)$ is the union of possible standard bases for $\mathscr{A}$.

We now set out to show that our class is a homogeneous MAEC.

Lemma 9.14. *$(\mathbb{K}_{\ell^p}, \preccurlyeq)$ is a metric abstract elementary class, i.e. satisfies Definition 2.1. Furthermore the Löwenheim-Skolem number LS^{d} of the class is $\aleph_0$.*

Proof. The first three items are trivial. For the fourth, note that if $\langle \mathscr{A}_i : i < \delta \rangle$ is an $\preccurlyeq$-increasing chain, we may write it as $\langle \ell^p(I_i) : i < \delta \rangle$ where $(I_i)_{i<\delta}$ is increasing. Then the completion of the union of the chain is

$$\ell^p \left(\bigcup_{i<\delta} I_i \right),$$

since each element of this space can be approximated by elements of finite support which hence are in some $\mathscr{A}_i$. This space also trivially satisfies demands (b) and (c) of the definition.

For the fifth item note that if $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}_{\ell^p}$, $\mathscr{A}, \mathscr{B} \preccurlyeq \mathscr{C}$ and $\mathscr{A} \subseteq \mathscr{B}$ then

$$\mathbb{B}(\mathscr{B}) = \mathbb{B}(\mathscr{C}) \cap \mathscr{B}$$

and (since $\mathscr{A} \subseteq \mathscr{B}$)

$$\mathbb{B}(\mathscr{A}) = \mathbb{B}(\mathscr{C}) \cap \mathscr{A} = \mathbb{B}(\mathscr{C}) \cap \mathscr{A} \cap \mathscr{B} = \mathbb{B}(\mathscr{B}) \cap \mathscr{A}$$

and by Remark 9.12 we are done.

Finally for the Löwenheim-Skolem number let $A \subseteq \ell^p(I)$ for some I and let $J = \mathrm{supp}(A) = \bigcup_{a \in A} \mathrm{supp}(a)$, the support of A. Then $|\{e_i : i \in J\}| \leq |A| \cdot \aleph_0$ and taking all rational linear combinations of these vectors only increases the cardinality by $\aleph_0$. Also these linear combinations are enough since the closure of them is $\ell^p(J)$ which is clearly the smallest $\preccurlyeq$-submodel of $\ell^p(I)$ containing A. □

Lemma 9.15. *If $\mathscr{A}, \mathscr{B} \in \mathbb{K}_{\ell^p}$ and $\mathscr{A} \cap \mathscr{B}$ is either empty or a $\preccurlyeq$-submodel of both $\mathscr{A}$ and $\mathscr{B}$, then there is $\mathscr{C} \in \mathbb{K}_{\ell^p}$ such that $\mathscr{A}, \mathscr{B} \preccurlyeq \mathscr{C}$. In particular $\mathbb{K}_{\ell^p}$ satisfies the joint embedding and amalgamation properties.*

Proof. Let $\mathscr{A}, \mathscr{B} \in \mathbb{K}_{\ell^p}$. Choose standard bases $I_{\mathscr{A}}$ for $\mathscr{A}$ and $I_{\mathscr{B}}$ for $\mathscr{B}$ such that the bases coincide in the set

$$\mathbb{B}(\mathscr{A}) \cap \mathbb{B}(\mathscr{B}).$$

Then the required model is

$$\mathscr{C} = \widetilde{\ell^p}(I_{\mathscr{A}} \cup I_{\mathscr{B}}).$$

□

Since we have amalgamation we now have a well-behaved notion of a Galois-type (see def. 2.8) and can turn our attention to the homogeneity property. First we take a closer look at the relation between embeddings and automorphisms.

Lemma 9.16. *Assume $(\mathbb{K}_{\ell^p}, \preccurlyeq)$ is the class defined in 9.10.*

(i) *If $\mathscr{A} \preccurlyeq \mathscr{B}$, $|\mathbb{B}(\mathscr{A})| < |\mathbb{B}(\mathscr{B})|$ and $f : \mathscr{A} \to \mathscr{B}$ is a $\mathbb{K}_{\ell^p}$-embedding then f extends to an isometric automorphism of $\mathscr{B}$.*

(ii) *If $a \in \mathscr{A}$ and $b \in \mathscr{B}$ are such that*

$$\mathrm{t}^g_{\mathscr{A}}(a/\emptyset) = \mathrm{t}^g_{\mathscr{B}}(b/\emptyset)$$

then there is $\mathscr{C}$ such that $\mathscr{A}, \mathscr{B} \preccurlyeq \mathscr{C}$ and $\mathscr{C}$ has an isometric automorphism mapping a to b.

Proof. For (i) let $\mathscr{A} \preccurlyeq \mathscr{B}$ and fix a standard basis $(\hat{e}_i)_{i \in J}$ of $\mathscr{B}$ and $I \subset J$ such that $(\hat{e}_i)_{i \in I}$ is a standard basis for $\mathscr{A}$. Now if $f : \mathscr{A} \to \mathscr{B}$ is a $\mathbb{K}_{\ell^p}$-embedding, $f((\hat{e}_i)_{i \in I}) \subset \mathbb{B}(\mathscr{B})$ and f is of the form

$$f(\sum_{i \in I} a_i \hat{e}_i) = \sum_{i \in I} \varepsilon_i a_i \hat{e}_{\sigma(i)}$$

for some one-to-one function $\sigma : I \to J$ and $\varepsilon_i \in \{-1, 1\}$. Since $|\mathbb{B}(\mathscr{A})| < |\mathbb{B}(\mathscr{B})|$, we also have $|I| < |J|$ and hence

$$|J \backslash I| = |J \backslash \sigma(I)|.$$

So σ can be extended to a permutation σ' of J and hence for any choice of $\varepsilon_i \in \{-1, 1\}$ for $i \notin I$, the function generated by

$$\hat{e}_i \mapsto \varepsilon_i \hat{e}_{\sigma'(i)}$$

is an isometric automorphism of $\mathscr{B}$ extending f.

For (ii) assume $a, b, \mathscr{A}, \mathscr{B}$ are as in the claim. By the definition of Galois-types there are $\mathscr{C}' \in \mathbb{K}_{\ell^p}$ and $\mathbb{K}_{\ell^p}$-embeddings $f : \mathscr{A} \to \mathscr{C}'$ and $g : \mathscr{B} \to \mathscr{C}'$ such that $f(a) = g(b)$. By 2.6 we may assume $g = \mathrm{id}$ and $\mathscr{B} \preccurlyeq \mathscr{C}'$. Now by 9.15 there is a model $\mathscr{C} \in \mathbb{K}_{\ell^p}$ such that $\mathscr{A}, \mathscr{C}' \preccurlyeq \mathscr{C}$ and we may assume $|\mathbb{B}(\mathscr{C})| > |\mathbb{B}(\mathscr{A})|$. Now, since

$$f(\mathscr{A}) \preccurlyeq \mathscr{C}' \preccurlyeq \mathscr{C},$$

f can be interpreted as a $\mathbb{K}_{\ell^p}$-embedding $\mathscr{A} \to \mathscr{C}$ and hence by (i) extends to an isometric automorphism of $\mathscr{C}$ mapping a to b. □

The following two lemmas are based on the same idea: when considering different ways of mapping a given element onto another, we only have finitely many alternatives for where an index of the support can be mapped.

Lemma 9.17. *For $1 \leq p < \infty$ and $p \neq 2$, $(\mathbb{K}_{\ell^p}, \preccurlyeq)$ is homogeneous.*

Proof. By Lemmas 9.15 and 9.16 it is enough to show the following:

If $(a_i)_{i<\alpha}, (b_i)_{i<\alpha} \subseteq \ell^p(J)$, $|J| > \alpha + \aleph_0$ and for all $n < \omega$ and $i_0, \dots, i_{n-1} < \alpha$ there is an isometric automorphism f of $\ell^p(J)$ such that

$$f(a_{i_k}) = b_{i_k} \text{ for each } k < n$$

then there is an isometric automorphism F of $\ell^p(J)$ such that

$$F(a_i) = b_i \text{ for each } i < \alpha.$$

So let $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ be as above. For $j \in J$ we denote by $a_i(j)$ the jth coordinate of a_i, i.e.

$$a_i = (a_i(j))_{j \in J} (\in \ell^p(J)).$$

Note that each element of $\ell^p(J)$ has at most countable support and that for each $c \in \ell^p(J)$ only finitely many of the coordinates $c(j), j \in J$, can have the same nonzero absolute value. Hence each element of $\ell^p(J)$ gives rise to an equivalence relation $\sim_c$ on J,

$$j_1 \sim_c j_2 \text{ if and only if } |c(j_1)| = |c(j_2)|.$$

which divides the support of c into finite equivalence classes. If c is a_i or b_i for some $i < \alpha$, we will name these equivalence classes

$$I^a_{(i,t)} = \{j \in J : |a_i(j)| = t\}, \qquad I^b_{(i,t)} = \{j \in J : |b_i(j)| = t\}$$

(if $\pm t$ is not in the range of a_i, $I^a_{(i,t)}$ is not an equivalence class but still a well-defined set, namely $\emptyset$).

Since by 9.11 an isometric automorphism of $\ell^p(J)$ is a combination of a permutation and changes of signs, it is clear that every isometric automorphism of $\ell^p(J)$ mapping a to b must map $I^a_{(i,t)}$ to the corresponding $I^b_{(i,t)}$. Furthermore by the assumption that finite tuples of $(a_i)_{i<\alpha}$ can be mapped by isometric automorphisms to the corresponding tuple from $(b_i)_{i<\alpha}$ it is clear that for all finite collections Γ of pairs (i,t), where $i<\alpha$ and $t \in \mathbb{R}_+$, the intersections

$$I^a_\Gamma = \bigcap_{(i,t)\in\Gamma} I^a_{(i,t)} \text{ and } I^b_\Gamma = \bigcap_{(i,t)\in\Gamma} I^b_{(i,t)} \tag{9.1}$$

have the same finite size (which of course may be 0). For a given $\Gamma \subset \alpha \times \mathbb{R}_+$ we say that the set I^a_Γ (or I^b_Γ respectively) is *minimal* if it is nonempty and for every $\Gamma' \supsetneq \Gamma$ either $I^a_{\Gamma'} = \emptyset$ or $I^a_{\Gamma'} = I^a_\Gamma$. Now since $I^a_{(i,t)}$ is finite for every $(i,t) \in \alpha\times\mathbb{R}_+$, if $I^a_{(i,t)}$ is nonempty there is a finite $\Gamma \ni (i,t)$ such that I^a_Γ is minimal. Furthermore the supports of $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ can be covered by (disjoint) minimal sets I^a_Γ, I^b_Γ where the sets Γ are finite.

Since $|\bigcup_{i<\alpha}(\operatorname{supp}(a_i)\cup\operatorname{supp}(b_i))| \le \alpha\cdot\aleph_0 < |J|$ it is clear that each partial mapping $\sigma : J \to J$ taking each I^a_Γ to the corresponding I^b_Γ can be extended to a permutation of J. We still need to show that we can find signs and fix the partial mappings on the minimal sets so that we get an isometric automorphism mapping each a_i to the corresponding b_i. So let $j \in J$ and fix a finite $\Gamma \subset \alpha\times\mathbb{R}_+$ such that I^a_Γ is minimal and contains j. For each $j \in I^a_\Gamma$ there is a unique partition P^a_j of α such that i_1 and i_2 belong to the same part if and only if $\operatorname{sign}(a_{i_1}(j)) = \operatorname{sign}(a_{i_2}(j))$, where

$$\operatorname{sign}(r) = \begin{cases} 1, \text{ if } r \ge 0, \\ 0, \text{ otherwise.}\end{cases}$$

Now we claim that there is a bijection $\sigma : I^a_\Gamma \to I^b_\Gamma$ such that

$$P^a_j = P^b_{\sigma(j)} \text{ for all } j \in I^a_\Gamma. \tag{9.2}$$

If not, let

$$S = \{\sigma : \sigma \text{ is a bijection } I^a_\Gamma \to I^b_\Gamma\}.$$

Since I^a_Γ is finite, S is finite. Now for each σ in S we can choose witnesses to the fact that for some $j \in I^a_\Gamma$ $P^a_j \ne P^b_{\sigma(j)}$, i.e. we choose $i^1_\sigma, i^2_\sigma \in I^a_\Gamma$ such that they are equivalent in P^a_j but $\sigma(i^1), \sigma(i^2)$ are nonequivalent in $P^b_{\sigma(j)}$ (or vice versa). Then

$$\mathscr{I} = \{i<\alpha : (i,t)\in\Gamma \text{ for some } t\} \cup \{i^l_\sigma < \alpha : \sigma\in S, l=1,2\}$$

is finite. By the assumption there should be an isometric automorphism of $\ell^p(J)$ mapping $(a_i)_{i\in\mathscr{I}}$ to $(b_i)_{i\in\mathscr{I}}$, but this is impossible since we know such an automorphism must map the basis element corresponding to I^a_Γ to the ones corresponding to I^b_Γ and we have collected witnesses spoiling every possible permutation doing this, a contradiction. Hence there must be some σ such that 9.2 holds. Then we let

$$\varepsilon_j = \begin{cases} 1, & \text{if } \operatorname{sign}(a_0(j)) = \operatorname{sign}(b_0(\sigma(j))), \\ -1, & \text{otherwise}\end{cases}$$

and are done. □

Remark 9.18. In addition to now having the properties necessary to build a monster model we actually know what the monster looks like. Since $\mathfrak{M} \in \mathbb{K}$ it is clear that it is an ℓ^p-space, but Lemma 9.17 shows that we do not need it to be especially large. For $\ell^p(I)$ to be μ-homogeneous it suffices that $|I| \ge \mu^+$. Also it is clear that any ℓ^p-space of cardinality $\le \mu$ is $\mathbb{K}_{\ell^p}$-embeddable into such a $\ell^p(I)$.

Lemma 9.19. *For $1 \leq p < \infty$, $p \neq 2$ and $\kappa > \omega$, $\ell^p(\kappa)$ satisfies the perturbation property 2.12.*

Proof. Let $(b_n)_{n<\omega}$ be a convergent sequence in $\ell^p(\kappa)$ of elements of a fixed type. Then for each $n < \omega$ there is an isometric automorphism f_n of $\ell^p(\kappa)$ mapping b_0 to b_n. Let $b = \lim_{n\to\infty} b_n$. We wish to show that $\mathrm{t}^g_{\ell^p(\kappa)}(b) = \mathrm{t}^g_{\ell^p(\kappa)}(b_n)$.

Denote by $b_n(i)$ the i'th element of b_n. Note that since all the b_n have the same type, their supports have the same size and also $\mathrm{rng}(b_n) = \{b_n(i) : i < \kappa\}$ is the same for every $n < \omega$ up to change of signs. Also, since $b_n \to b$ as $n \to \infty$ we have for each $i < \kappa$, $b_n(i) \to b(i)$ as $n \to \infty$. Further $\mathrm{rng}(b)$ is nowhere dense, i.e. inside every interval there is an interval not intersecting $\mathrm{rng}(b)$ since else there would be some $\varepsilon > 0$ with $b(i) > \varepsilon$ for infinitely many i, which is impossible. Hence for every n there are p_n, δ_n with $0 < p_n < \frac{1}{n+1}$ such that

$$[p_n - \delta_n, p_n + \delta_n] \cap \mathrm{rng}(b) = \emptyset \text{ and } [-p_n - \delta_n, -p_n + \delta_n] \cap \mathrm{rng}(b) = \emptyset.$$

Now for each $n < \omega$, there are only finitely many indices i for which $|b(i)| > p_n$. Hence the sets

$$\begin{aligned} I_0 &= \{i < \kappa : |b(i)| > p_0\}, \\ I_{n+1} &= \{i < \kappa : p_n > |b(i)| > p_{n+1}\} \end{aligned}$$

are finite. Since $b_n \to b$ there is, for each $n < \omega$, some $N_n < \omega$ such that for all $m \geq N_n$, $p_{n-1} > |b_m(i)| > p_n$ (or just $|b_m(i)| > p_n$ if $n = 0$) if and only if $i \in I_n$. Hence from N_n onwards there are only finitely many possible values for $b_m(i)$ for $i \in I_n$, so from some N'_n onwards the sequence $(b_m(i))_{N'_n<m<\omega}$ is constant for each $i \in I_n$. Now denote

$$A_n = \{f^{-1}_{N'_n}(e_i) : i \in I_n\}$$

and note that if $m \geq N'_n$ and $e \in A_n$ then $f_m(e) \in \{e_i : i \in I_n\}$ so the sets A_n are disjoint. Also $\bigcup_{n<\omega} = A_n = \mathrm{supp}(b_0)$ since for every $i \in \mathrm{supp}(b_0)$ there is $n < \omega$ such that $p_{n-1} > |b_0(i)| > p_n$ (or $|b_0(i)| > p_0$) and then $f_{N'_n}$ will map e_i into $\{e_j : j \in I_n\}$. Now let g map $\{e_j : j \in \kappa \backslash \mathrm{supp}(b_0)\}$ to $\{e_j : j \in \kappa \backslash \mathrm{supp}(b)\}$, the function

$$\bigcup_{n<\omega} f_{N'_n} \restriction A_n \cup g$$

induces an isometric automorphism of $\ell^p(\kappa)$ mapping b_0 to b and hence proving that b is of the desired type. □

We finally have the theorem we have been working towards.

Theorem 9.20. *For $1 \leq p < \infty$, $p \neq 2$, the class of Banach spaces isometrically isomorphic to some $\ell^p(I)$ with $\preccurlyeq$ defined as in (9.10) forms a homogeneous metric abstract elementary class with Löwenheim-Skolem number $\aleph_0$.*

Proof. This was proved in lemmas 9.14 to 9.19, since the class trivially contains arbitrary large models. □

Trivially we see that the class ℓ^p-spaces is categorical in every $\kappa^{\aleph_0} = \kappa$ satisfying $\xi^{\aleph_0} < \kappa$ for every $\xi < \kappa$.

10 Metric homogeneity

This final section introduces the notion of *metric homogeneity*. It enables us to prove categoricity transfer from any uncountable κ to cardinals greater than some $\xi < \beth_{\mathfrak{c}^+}$.

Definition 10.1. $\mathbb{K}$ is *metricly homogeneous* if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all finite $a, b \in \mathfrak{M}$ and all sets A, if $\mathbf{d}(\mathrm{t}^g(a/A), \mathrm{t}^g(b/A)) > \varepsilon$ then there is a finite $A' \subset A$ such that $\mathbf{d}(\mathrm{t}^g(a/A'), \mathrm{t}^g(b/A')) > \delta$.

10.1 Better tools enabled

In this section we prove that metric homogeneity gives us stability transfer and saturated models. We also show that the property holds in $\mathbb{K}_{\ell^p}$, our example class of ℓ^p-spaces.

Notation 10.2. If $\mathbb{K}$ is metricly homogeneous and $\varepsilon > 0$, we denote the $\delta > 0$ from the definition by $f^{MH}(\varepsilon)$.

Remark 10.3. Note that $f^{MH}(\varepsilon) \leq \varepsilon$.

To prove stability transfer we need some ε-splitting calculations.

Lemma 10.4. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable and metricly homogeneous. Assume further that $A \subset \mathscr{A} \subset B$, A is countable, $\mathscr{A}$ realizes all types over finite subsets of A, $\mathbf{d}(\mathrm{t}^g(a/\mathscr{A}), \mathrm{t}^g(b/\mathscr{A})) = \delta_1$, $a \downarrow^{\delta_2}_A B$ and $b \downarrow^{\delta_3}_A B$. Then if*

$$\delta_1 + \delta_2 + \delta_3 \leq f^{MH}(\varepsilon),$$

we have

$$\mathbf{d}(\mathrm{t}^g(a/B), \mathrm{t}^g(b/B)) \leq \varepsilon.$$

Note that one or some of the δ's may be 0.

Proof. Assume towards a contradiction that $\mathbf{d}(\mathrm{t}^g(a/B), \mathrm{t}^g(b/B)) > \varepsilon$. By metric homogeneity, let $C \subset B$ be finite such that $\mathbf{d}(\mathrm{t}^g(a/C), \mathrm{t}^g(b/C)) = d > f^{MH}(\varepsilon)$ and let $0 < \delta' < d - f^{MH}(\varepsilon)$. Let $A' \subset A$ be finite and such that $a \downarrow^{\delta_2+\delta'/2}_{A'} B$ and $b \downarrow^{\delta_3+\delta'/2}_{A'} B$. Let $f \in \mathrm{Aut}(\mathfrak{M}/A')$ be such that $f(C) \subset \mathscr{A}$. Now

$$\begin{aligned} d &= \mathbf{d}(\mathrm{t}^g(a/C), \mathrm{t}^g(b/C)) \\ &= \mathbf{d}(\mathrm{t}^g(f(a)/f(C)), \mathrm{t}^g(f(b)/f(C))) \\ &\leq \mathbf{d}(\mathrm{t}^g(f(a)/f(C)), \mathrm{t}^g(a/f(C))) \\ &\quad + \mathbf{d}(\mathrm{t}^g(a/f(C)), \mathrm{t}^g(b/f(C))) \\ &\quad + \mathbf{d}(\mathrm{t}^g(b/f(C)), \mathrm{t}^g(f(b)/f(C))) \\ &\leq \delta_2 + \delta'/2 + \delta_1 + \delta_3 + \delta'/2 \\ &\leq f^{MH}(\varepsilon) + \delta' \\ &< d, \end{aligned}$$

a contradiction. ∎

Lemma 10.5. *If $a \downarrow^0_A B$ and $\mathrm{d}(b, a) < \delta$ then $b \downarrow^{2\delta}_A B$.*

Proof. Let $\varepsilon = \delta - \mathrm{d}(b,a) > 0$. Let $A' \subset A$ be finite and such that $a \downarrow^{\varepsilon}_{A'} B$. We claim that $b \downarrow^{2\delta}_{A'} B$. Assume towards a contradiction that $c, d \in B$ and $f \in \mathrm{Aut}(\mathfrak{M}/A')$ are such that $f(d) = c$ and $\mathbf{d}(\mathrm{t}^g(b/A'c), \mathrm{t}^g(f(b)/A'c)) \geq 2\delta$. Then

$$\begin{aligned} 2\delta &\leq \mathbf{d}(\mathrm{t}^g(b/A'c), \mathrm{t}^g(f(b)/A'c)) \\ &\leq \mathbf{d}(\mathrm{t}^g(b/A'c), \mathrm{t}^g(a/A'c)) \\ &\quad + \mathbf{d}(\mathrm{t}^g(a/A'c), \mathrm{t}^g(f(a)/A'c)) \\ &\quad + \mathbf{d}(\mathrm{t}^g(f(a)/A'c), \mathrm{t}^g(f(b)/A'c)) \\ &\leq \mathrm{d}(b,a) + \varepsilon + \mathrm{d}(a,b) \\ &< 2\delta, \end{aligned}$$

a contradiction. □

We now turn our attention to stability transfer. The proof is done in parts due to the nontrivialness of extending sets to $F^{\mathfrak{M}}_{\omega}$-saturated models.

Lemma 10.6. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable and metricly homogeneous. If for all B with $|B| = \lambda$ there exists an $F^{\mathfrak{M}}_{\omega}$-saturated $\mathscr{B} \supset B$ with $|\mathscr{B}| = \lambda$ then $\mathbb{K}$ is λ-$\mathbf{d}$-stable.*

Proof. Assume towards a contradiction that $|B| = \lambda$, B is $F^{\mathfrak{M}}_{\omega}$-saturated but $|S(B)| > \lambda$. Then there are some $\varepsilon > 0$ and types $p_i \in S(B)$, for $i < \lambda^+$ such that $\mathbf{d}(p_i, p_j) > \varepsilon$ for all $i < j < \lambda^+$. By ω-$\mathbf{d}$-stability and lemma 4.5 choose for each $i < \lambda^+$ a finite $A_i \subset B$ such that p_i does not $f^{MH}(\varepsilon)/3$-split over A_i. λ^+ many of these A_i's are the same, denote it by A and re-enumerate the types for which $A_i = A$. Next extend A to $\mathscr{A} \subset B$ realizing all types over A: Let $A_0 = A$ and for each $n < \omega$ let A_{n+1} contain A_n and realize a countable dense subset of $S(A_n)$. Then define $\mathscr{A} = \overline{\bigcup_{n<\omega} A_n}$. Since B is $F^{\mathfrak{M}}_{\omega}$-saturated we see by a construction similar to that in theorem 3.7 that $\mathscr{A}$ in fact does realize all $p \in S(A)$.

Now for $i < j < \lambda^+$, since $\mathbf{d}(p_i, p_j) > \varepsilon$, by metric homogeneity there is some finite $C \in B$ such that

$$\mathbf{d}(p_i \restriction C, p_j \restriction C) > f^{MH}(\varepsilon).$$

Let $a, b \in \mathfrak{M}$ realize p_i, p_j respectively. Since $\mathscr{A}$ realizes $\mathrm{t}^g(C/A)$ there is some $f \in \mathrm{Aut}(\mathfrak{M}/A)$ with $f(C) = C' \in \mathscr{A}$. Then

$$\begin{aligned} f^{MH}(\varepsilon) &< \mathbf{d}(\mathrm{t}^g(a/C), \mathrm{t}^g(b/C)) \\ &= \mathbf{d}(\mathrm{t}^g(f(a)/C'), \mathrm{t}^g(f(b)/C')) \\ &= \mathbf{d}(\mathrm{t}^g(f(a)/C'), \mathrm{t}^g(a/C')) \\ &\quad + \mathbf{d}(\mathrm{t}^g(a/C'), \mathrm{t}^g(b/C')) \\ &\quad + \mathbf{d}(\mathrm{t}^g(b/C'), \mathrm{t}^g(f(b)/C')) \\ &\leq 2f^{MH}(\varepsilon)/3 + \mathbf{d}(\mathrm{t}^g(a/C'), \mathrm{t}^g(b/C')). \end{aligned}$$

Hence $\mathbf{d}(\mathrm{t}^g(a/\mathscr{A}), \mathrm{t}^g(b/\mathscr{A})) \geq \mathbf{d}(\mathrm{t}^g(a/C'), \mathrm{t}^g(b/C')) \geq f^{MH}(\varepsilon)/3$ and since we can do the same conclusion for all $i < j < \lambda^+$ this gives us a type-space of density $\geq \lambda^+$ over a separable set, a contradiction. □

Lemma 10.7. *Assume $\mathbb{K}$ is λ-$\mathbf{d}$-stable. Then if $|B| \geq \lambda^+$, there exists an $F^{\mathfrak{M}}_{\lambda^+}$-saturated $\mathscr{B} \supset B$ with $|\mathscr{B}| \geq \lambda^+$.*

Proof. Let B with $|B| \leq \lambda^+$ be given. Fix some dense $B' \subset B$ of cardinality $\leq \lambda^+$ and enumerate $B' = \{b_i : i < \lambda^+\}$. Then construct B_i, for $i < \lambda^+$, inductively as follows:

- $B_0 = \emptyset$,
- B_{i+1} contains $B_i \cup \{b_i\}$ and realizations for a dense subset of $S(B_i)$,
- for δ a limit $B_\delta = \bigcup_{i<\delta} B_i$.
- $|B_i| \leq \lambda$.

In the end let $\mathscr{B} = \overline{\bigcup_{i<\lambda^+} B_i}$. Then $|\mathscr{B}| \leq \lambda^+$ and since $\mathscr{B} \supset B'$ and is (metricly) closed, it contains B. To show that $\mathscr{B}$ is $F^{\mathfrak{M}}_{\lambda^+}$-saturated, by theorem 3.7 it is enough to show that $\mathscr{B}$ is $F^{\mathfrak{M}}_{\lambda^+}$-$\mathbf{d}$-saturated. So let $C \subset \mathscr{B}$, $|C| < \lambda^+$, $a \in \mathfrak{M}$ and $\varepsilon > 0$ be given. Let $C' \subset C$ be a dense subset of cardinality $< \lambda^+$. Now for each $c \in C'$ there is some countable $C_c \subset \bigcup_{i<\lambda^+} B_i$ such that $c \in \overline{C_c}$. Let $C^+ = \bigcup_{c \in C'} C_c$. Then $C^+ \subset \bigcup_{i<\lambda^+} B_i$ and C^+ has cardinality $< \lambda^+$ so $C^+ \subset B_i$ for some $i < \lambda^+$. Hence there is some $a' \in B_{i+1}$ with $\mathbf{d}(\mathrm{t}^g(a'/B_i), \mathrm{t}^g(a/B_i)) < \varepsilon$ and by the way C^+ was chosen, the type over B_i determines the type over C so we are done. □

Lemma 10.8. *Assume $\mathbb{K}$ is metricly homogeneous, $\lambda > \aleph_0$ is a limit cardinal and that for all infinite $\xi < \lambda$, $\mathbb{K}$ is ξ-$\mathbf{d}$-stable (especially ω-$\mathbf{d}$-stable) and for any set B of density $\xi > \aleph_0$ there exists an $F^{\mathfrak{M}}_{\xi}$-saturated $\mathscr{B} \supset B$ of density ξ. Then any set B of density λ can be extended to some $F^{\mathfrak{M}}_{\lambda}$-saturated $\mathscr{B} \supset B$ of density λ.*

Proof. Let B of density λ be given and let $B' \subset B$ be a dense subset of cardinality λ. Enumerate $B' = \{b_i : i < \lambda\}$. Let $\kappa = \mathrm{cf}(\lambda)$ and choose an increasing sequence of cardinals λ_i, for $i < \kappa$ such that $\aleph_1 \leq \lambda_i < \lambda$. We construct B_i by induction on $i < \kappa$ as follows:

- B_0 is $F^{\mathfrak{M}}_{\aleph_1}$-saturated,
- B_{i+1} contains B_i, $\{b_j : j < \lambda_i\}$ and a length-λ_n Morley sequence of realizations for each type in a dense subset of $S(B_n)$, is $F^{\mathfrak{M}}_{\lambda_i}$-saturated and of size λ_n,
- for limit δ, $B_\delta = \bigcup_{i<\delta} B_i$.

We build the Morley sequences as in remark 8.4, by extending the set to an $F^{\mathfrak{M}}_{\aleph_1}$-saturated model after adding a new element to the sequence. Note that this does not increase the size of B_i. In the end let $\mathscr{B} = \overline{\bigcup_{i<\kappa} B_i}$. To prove that $\mathscr{B}$ is $F^{\mathfrak{M}}_{\lambda}$-saturated, we prove $F^{\mathfrak{M}}_{\lambda}$-$\mathbf{d}$-saturation. So let $C \subset \mathscr{B}$, $|C| < \lambda$, $a \in \mathfrak{M}$ and $\varepsilon > 0$. As in lemma 10.7 we can find $C^+ \subset \bigcup_{i<\kappa} B_i$ which is dense in C and of cardinality $\aleph_0 \cdot |C| < \lambda$. By lemma 4.5 there is $i' < \kappa$ such that

$$a \downarrow^{f^{MH}(\varepsilon)/2}_{B_{i'}} \bigcup_{j<\kappa} B_j.$$

Then we can choose $i \geq i'$ such that $\lambda_i > \mathrm{card}(C^+)$. Choose $p \in S(B_i)$ such that $\mathbf{d}(p, \mathrm{t}^g(a/B_i)) < f^{MH}(\varepsilon)/2$ and there is a Morley sequence $(a_j)_{j<\lambda_i}$ in B_{i+1} realizing p. We claim that for some $j < \lambda_i$, $a_j \downarrow_{B_i} B^+$. Otherwise, by finite character, for each $j < \lambda_i$, there exists some $c_j \in B^+$ such that $a_j \not\downarrow_{B_i} c_j$, and since $\lambda_i > \mathrm{card}(B^+)$, λ_i many c_j's are the same, denote it c. But then there is a λ_i-sequence of a_j's such that $a_j \not\downarrow_{B_i} c$.

Then if D_j are the $F^{\mathfrak{M}}_{\aleph_1}$-saturated models from the Morley sequence construction, with $D_j \supset \{a_{j'} : j' < j\}$, we have

$$a_j \downarrow_{B_i} D_j \text{ and } a_j \not\downarrow_{B_i} c$$

so by transitivity we have

$$a_j \not\downarrow_{D_j} c,$$

i.e. by symmetry and monotonicity

$$c \not\downarrow_{D_j} D_{j+1}.$$

But by [HS00, Lemma 3.2(iii)] this gives a strongly splitting sequence of length $\lambda_i \geq \aleph_1$, a contradiction. Hence there must be some a_j satisfying $a_j \downarrow_{B_i} B^+$. Since B_i is (at least) $F^{\mathfrak{M}}_{\aleph_1}$-saturated, by theorem 6.4 $a_j \downarrow^0_{B_i} B^+$.

Now $\mathbf{d}(\mathrm{t}^g(a_j/B_i), \mathrm{t}^g(a/B_i)) < f^{MH}(\varepsilon)/2$, $a_j \downarrow^0_{B_i} B^+$ and $a \downarrow^{f^{MH}(\varepsilon)/2}_{B_{i'}} \bigcup_{j<\kappa} B_j$, especially $a \downarrow^{f^{MH}(\varepsilon)/2}_{B_i} B^+$, so by lemma 10.4 $\mathbf{d}(\mathrm{t}^g(a_j/B^+), \mathrm{t}^g(a/B^+)) \leq \varepsilon$ and we are done. □

The previous three lemmas give us the following theorems:

Theorem 10.9. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable and metricly homogeneous, then it is λ-$\mathbf{d}$-stable for all λ.*

Theorem 10.10. *If $\mathbb{K}$ is ω-$\mathbf{d}$-stable and metricly homogeneous and $\lambda > \aleph_0$ then every set B of density $\leq \lambda$ can be extended to an $F^{\mathfrak{M}}_{\lambda}$-saturated $\mathscr{B} \supset B$ of density λ.*

Theorem 10.11. *$\mathbb{K}_{\ell^p}$ is metricly homogeneous with $f^{MH} = \mathrm{id}$.*

Proof. Fix finite tuples a and b and a set A such that for all finite $B \subset A$,

$$\mathbf{d}(\mathrm{t}^g(a/B), \mathrm{t}^g(b/B)) \leq \varepsilon.$$

We wish to show that $\mathbf{d}(\mathrm{t}^g(a/A), \mathrm{t}^g(b/A)) \leq \varepsilon$, i.e. for all $\delta > 0$ find an automorphism f of the monster fixing A pointwise and satisfying $\mathrm{d}(f(a), b) \leq \varepsilon + \delta$.

Recall that $\mathfrak{M}$ is just some $\ell^p(I)$ with $|I| > |A|$ and that the (isometric) automorphisms of $\ell^p(I)$ are of the form

$$U(\sum_{i\in I} x_i e_i) = \sum_{i \in I} \epsilon_i x_i e_{\sigma(i)},$$

where $(e_i)_{i\in I}$ is the standard basis of $\ell^p(I)$, σ is a permutation of I and $\epsilon_i \in \{-1, 1\}$ for each $i \in I$. Hence, as in the proof of lemma 9.17, the demand that f fix A pointwise gives rise to a partition of $\mathrm{supp}(A) = \bigcup\{\mathrm{supp}(c) : c \in A\}$ with finite equivalence classes corresponding to the absolute values of the coordinates of elements of A. Since a and b are finite, the union of their support is countable and of $\mathrm{supp}(A)$ we only need to consider those equivalence classes that intersect $\mathrm{supp}(a) \cup \mathrm{supp}(b)$. We can enumerate these as $(I_n)_{n<\omega}$. In addition to these we consider the part of $\mathrm{supp}(a) \cup \mathrm{supp}(b)$ outside $\bigcup_{n<\omega} I_n$, denote it by J, and a countable set $J_\emptyset$ intersecting neither $\mathrm{supp}(A)$ nor $\mathrm{supp}(a) \cup \mathrm{supp}(b)$. Hence we restrict our attention to the countable set $I' = \bigcup_{n<\omega} I_n \cup J \cup J_\emptyset$. Outside this index set f can be built of the identity permutation. Also, when by the assumption we are given an automorphism g of the monster fixing some finite $B \subset A$, we may assume that it permutes I' and $I \backslash I'$ separately, since changing the permutation associated to g (i.e. permuting the coordinates in $g(a)$) outside $\mathrm{supp}(b)$ does not affect the distance $\mathrm{d}(g(a), b)$ and $J_\emptyset$ gives us space enough inside I' to adjust the permutation.

Inside each I_n, consider the set S_n of all pairs of permutations and tuples of signs giving rise to (part of) an automorphism fixing A:

$$S_n = \{(\sigma, (\epsilon_i)_{i\in I_n}) : \sigma \text{ a permutation of } I_n,\ \epsilon_i \in \{-1,1\},$$
$$\sum_{i\in I_n} a_i e_i = \sum_{i\in I_n} \epsilon_i a_i e_{\sigma(i)} \text{ for all } (a_i)_{i\in I} \in A\}.$$

Since I_n is finite we only need a finite subset of A to rule out automorphisms whose permutation or signs do not match the configurations in S_n. Hence, for each $n < \omega$ there is an automorphism g of $\ell^p(I)$ matching some configuration in S_m for each $m \leq n$ and also satisfying

$$\sum_{i\in I} |(g(a^k))_i - b^k_i|^p = \sum_{i\in I} |\epsilon^g_i a^k_i - b^k_{\sigma(i)}|^p \leq \varepsilon^p, \tag{10.1}$$

for each $k < \text{length}(a)$, where a^k and b^k denote the coordinates of the finite tuples a and b.

Now we turn to the task of finding suitable automorphisms for given δ's. So let $\delta > 0$ be given. Let $\delta' = \delta/4$ and choose $n < \omega$ such that for each $k < \text{length}(a)$,

$$\Big(\sum_{i\in\bigcup_{m>n} I_m} |a^k_i|^p\Big)^{\frac{1}{p}} < \delta' \text{ and } \Big(\sum_{i\in\bigcup_{m>n} I_m} |b^k_i|^p\Big)^{\frac{1}{p}} < \delta'. \tag{10.2}$$

By the assumption and the considerations when defining I' there is a permutation σ of I' and signs $\epsilon_i \in \{-1,1\}$ for $i \in I'$ such that

$$\Big(\sum_{i\in I'} |\epsilon_i a^k_i - b^k_{\sigma(i)}|^p\Big)^{\frac{1}{p}} \leq \varepsilon \tag{10.3}$$

for each $k < \text{length}(a)$. Now define a new permutation σ' such that

- $\sigma' \restriction \bigcup_{m\leq n} I_m = \sigma \restriction \bigcup_{m\leq n} I_m$,
- $\sigma' \restriction \bigcup_{m>n} I_m = \text{id}$,
- if $i, \sigma(i) \in J$ then $\sigma'(i) = \sigma(i)$ otherwise for $i \in J$, let $\sigma'(i) \in J_\emptyset$ and finally $\sigma' \restriction J_\emptyset$ is defined such that $\sigma' \restriction (J \cup J_\emptyset)$ becomes a permutation.

Now we observe that

- for $i \in \bigcup_{m\leq n} I_m$, $|\epsilon_i a^k_i - b^k_{\sigma'(i)}| = |\epsilon_i a^k_i - b^k_{\sigma(i)}|$,
- for $i \in \bigcup_{m>n} I_m$, $|a^k_i - b^k_{\sigma'(i)}| = |a^k_i - b^k_i| \leq |a^k_i| + |b^k_i|$,
- if $i \in J \cup J_\emptyset$ we have four cases:
  - $i, \sigma'(i) \in J$: then $\sigma'(i) = \sigma(i)$,
  - $i \in J$ and $\sigma'(i) \in J_\emptyset$: then either $\sigma(i) \in \bigcup_{m>n} I_m$ or $\sigma(i) \in J_\emptyset$ and

$$\begin{aligned} |\epsilon_i a^k_i - b^k_{\sigma'(i)}| &\leq |\epsilon_i a^k_i| + |b^k_{\sigma'(i)}| \\ &= |\epsilon_i a^k_i| + 0 \\ &\leq |\epsilon_i a^k_i - b^k_{\sigma(i)}| + |b^k_{\sigma(i)}|, \end{aligned}$$

– $i \in J_\emptyset$ and $\sigma'(i) \in J$: then $\sigma'(i) = \sigma(j)$ for some j either in $\bigcup_{m>n} I_m$ or in $J_\emptyset$ and

$$\begin{aligned} |\epsilon_i a_i^k - b_{\sigma'(i)}^k| &\leq |a_i^k| + |b_{\sigma'(i)}^k| \\ &= 0 + |b_{\sigma'(i)}^k| \\ &= |b_{\sigma(j)}^k| \\ &\leq |\epsilon_j a_j^k - b_{\sigma(j)}^k| + |\epsilon_j a_j^k|, \end{aligned}$$

– $i, \sigma'(i) \in J_\emptyset$: then $|\epsilon_i a_i^k - b_{\sigma'(i)}^k| = 0$.

Now let $\epsilon_i' = \epsilon_i$ for $i \notin \bigcup_{m>n} I_m$ and 1 for $i \in \bigcup_{m>n} I_m$. Then combining the above observations with (10.2) and (10.3) and using Minkowski's inequality, we obtain

$$\begin{aligned} \left(\sum_{i\in I'} |\epsilon_i' a_i^k - b_{\sigma'(i)}^k|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i\in I'} |\epsilon_i a_i^k - b_{\sigma(i)}^k|^p\right)^{\frac{1}{p}} \\ &\quad +2\left(\sum_{i\in\bigcup_{m>n} I_m} |a_i^k|^p\right)^{\frac{1}{p}} \\ &\quad +2\left(\sum_{i\in\bigcup_{m>n} I_m} |b_i^k|^p\right)^{\frac{1}{p}} \\ &\leq \varepsilon + 4\delta' \\ &= \varepsilon + \delta. \end{aligned}$$

Then σ' and the signs ϵ_i' (together with the identity mapping outside I') give rise to an automorphism g of the monster fixing A pointwise and satisfying $\mathrm{d}(g(a), b) \leq \varepsilon + \delta$. ☐

10.2 Primary models

In this section we define new isolation and saturation notions and build primary models. We also prove an ε-version of dominance for this isolation notion.

10.2.1 $[B]$-isolation

Definition 10.12. We say that $\mathrm{t}^g(a/A)$ is *ε-isolated with respect to* $B \subset A$ (*$[B]$-ε-isolated* for short) if there exist some $\delta > 0$ and finite $A_\varepsilon \subset A$ such that for all $f \in \mathrm{Aut}(\mathfrak{M}/(A_\varepsilon \cap B))$ with $\mathrm{d}(f(c), c) < \delta$ for all $c \in A_\varepsilon$ and for all finite $A^+ \subset A$, there exists $f' \in \mathrm{Aut}(\mathfrak{M}(A^+ \cap B))$ with $\mathrm{d}(f'(c), c) < \varepsilon$ for all $c \in A^+$ and $\mathrm{d}(f'(a), f(a)) < \varepsilon$ (in other words $\mathbf{d}(\mathrm{t}^g(f(a)A^+/A^+ \cap B), \mathrm{t}^g(aA^+/A^+ \cap B)) < \varepsilon$).

Definition 10.13. $\mathrm{t}^g(a/A)$ is *isolated with respect to* $B \subset A$ (*$[B]$-isolated* for short) if it is $[B]$-ε-isolated for all $\varepsilon > 0$.

Notation 10.14. If $\delta > 0$ and $A_\varepsilon \subset A$ are as in the definitions above, we say that δ, A_ε ε-isolate $\mathrm{t}^g(a/A)$ with respect to B.

Notation 10.15. We write $\mathrm{t}^g(a/A) =^\delta_{[B]} \mathrm{t}^g(b/A)$ if there exists $f \in \mathrm{Aut}(\mathfrak{M}/A \cap B)$ with $\mathrm{d}(f(c), c) < \delta$ for all $c \in A$ and $f(a) = b$.

Remark 10.16. The above is not an equivalence relation. However, it is symmetric and reflexive and instead of transitivity we have: if $\mathrm{t}^g(a/A) =^{\delta_1}_{[B]} \mathrm{t}^g(b/A)$ and $\mathrm{t}^g(b/A) =^{\delta_2}_{[B]} \mathrm{t}^g(c/A)$, then $\mathrm{t}^g(a/A) =^{\delta_1+\delta_2}_{[B]} \mathrm{t}^g(c/A)$.

Lemma 10.17. *Assume $B \subset A$, $(A_n)_{n<\omega}$ is an increasing sequence of finite sets with $A_n \subset A$, $\delta_n > 0$, for $n < \omega$ are such that $\sum_{n<\omega} \delta_n < \infty$ and a_n, for $n < \omega$ satisfy*

$$\mathrm{t}^g(a_{n+1}/A_n) =^{\delta_n}_{[B]} \mathrm{t}^g(a_n/A_n).$$

Then there is a such that

$$\mathrm{t}^g(a/A_n) =^{\varepsilon_n}_{[B]} \mathrm{t}^g(a_n/A_n),$$

where $\varepsilon_n = \sum_{i=n}^{\infty} \delta_i$.

Proof. Let $B \subset A$, A_n, a_n and δ_n be as above. Then there are $f_n \in \mathrm{Aut}(\mathfrak{M})$, for $n < \omega$, such that

- $f_o = \mathrm{id}$,
- $f_{n+1}(a_{n+1}) = a_n$,
- $f_{n+1} \restriction B \cap A_n = \mathrm{id}$ and $\mathrm{d}(f_{n+1}(c), c) < \delta_n$ for all $c \in A_n$.

By now defining $F_0 = f_0 = \mathrm{id}$ and $F_{n+1} = F_n \circ f_{n+1}$, we obtain $F_n \in \mathrm{Aut}(\mathfrak{M})$, for $n < \omega$, satisfying:

- $F_{n+1}(a_{n+1}) = a_0$,
- for all $c \in A_n$, $\mathrm{d}(F_{n+1}(c), F_n(c)) = \mathrm{d}(F_n \circ f_{n+1}(c), F_n(c)) = \mathrm{d}(f_{n+1}(c), c) < \delta_n$,
- for all $c \in A_n \cap B$, $\mathrm{d}(F_{n+1}(c), F_n(c)) = \mathrm{d}(f_{n+1}(c), c) = 0$.

Because $\sum_{n<\omega} \delta_n$ converges, the sequence $(F_n \restriction (\bigcup_{m<\omega} A_m))_{n<\omega}$ converges (pointwisely), so by the perturbation property there is $F \in \mathrm{Aut}(\mathfrak{M})$ such that for all $n < \omega$ and all $c \in A_n$, $\mathrm{d}(F(c), F_n(c)) \leq \varepsilon_n$ and for all $c \in A_n \cap B$, $\mathrm{d}(F(c), F_n(c)) = 0$. Hence, we can define $a = F^{-1}(a_0)$, and obtain, for all $n < \omega$,

- $F_n^{-1} \circ F(a) = a_n$,
- for all $c \in A_n$, $\mathrm{d}(F_n^{-1} \circ F(c), c) = \mathrm{d}(F(c), F_n(c)) \leq \varepsilon_n$,
- $F_n^{-1} \circ F \restriction A_n \cap B = \mathrm{id}$,

i.e. $\mathrm{t}^g(a/A_n) =^{\varepsilon_n}_{[B]} \mathrm{t}^g(a_n/A_n)$. □

Lemma 10.18. *Assume $\mathbb{K}$ is ω-**d**-stable. Then for all a, $B \subset A$, finite $C \subset A$, $\varepsilon > 0$ and $\delta > 0$ there exist a', $\delta_\varepsilon > 0$ and a finite $A_\varepsilon \subset A$ with $A_\varepsilon \supset C$ such that $\mathrm{t}^g(a'/C) =^{\delta}_{[B]} \mathrm{t}^g(a/C)$ and δ_ε, A_ε ε-isolate $\mathrm{t}^g(a'/A)$ with respect to B.*

Proof. Assume towards a contradiction that for all a' with $\mathrm{t}^g(a'/C) =^{\delta}_{[B]} \mathrm{t}^g(a/C)$ and all $\delta_\varepsilon > 0$ and finite $\mathscr{A}_\varepsilon$ there is $f \in \mathrm{Aut}(\mathfrak{M}/A_\varepsilon \cap B)$, with $\mathrm{d}(f(c), c) < \delta_\varepsilon$ for all $c \in A_\varepsilon$, but for some finite $A^+ \subset A$, $\mathbf{d}(\mathrm{t}^g(A^+a'/A^+ \cap B), \mathrm{t}^g(A^+f(a')/A^+ \cap B)) \geq \varepsilon$. Then we can define δ_n, A_n, for $n < \omega$, and a_ξ, f_ξ, A_ξ, for $\xi \in {}^{<\omega}2$, such that

- $\delta_n = 2^{-(n+1)}\delta'$, where $\delta' = \min\{\delta, \varepsilon/2\}$,
- $A_0 = C$, $A_n \subset A_{n+1} \subset A$, A_n is finite,
- $a_\emptyset = a$, $\mathrm{t}^g(a_\xi/C) =^{\delta}_{[B]} \mathrm{t}^g(a/C)$,
- $f_\emptyset = \mathrm{id}$ and for all $\xi \in {}^{<\omega}2$, $f_{\xi^\frown(0)} = \mathrm{id}$,
- $f_{\xi^\frown(1)}$ and the finite set $A_\xi \supset A_{\mathrm{length}(\xi)}$ are such that $f_{\xi^\frown(1)} \in \mathrm{Aut}(\mathfrak{M}/A_{\mathrm{length}(\xi)} \cap B)$, $\mathrm{d}(f_{\xi^\frown(1)}(c), c) < \delta_{\mathrm{length}(\xi)}$ for all $c \in A_{\mathrm{length}(\xi)}$, and
$$\mathbf{d}(\mathrm{t}^g(A_\xi a_\xi/A_\xi \cap B), \mathrm{t}^g(A_\xi f_{\xi^\frown(1)}(a_\xi)/A_\xi \cap B)) \geq \varepsilon,$$
- $a_{\xi^\frown(i)} = f_{\xi^\frown(i)}(a_\xi)$ for $i \in \{0,1\}$,
- $A_{n+1} = \bigcup_{\mathrm{length}(\xi)=n} A_\xi$.

Now $\mathrm{t}^g(f_{\xi^\frown(i)}(a_\xi)/A_{\mathrm{length}(\xi)}) =^{\delta_{\mathrm{length}(\xi)}}_{[B]} \mathrm{t}^g(a_\xi/A_{\mathrm{length}(\xi)})$, so since $\sum_{n=0}^{\mathrm{length}(\xi)-1} \delta_n < \delta' \leq \delta$, we have $\mathrm{t}^g(a_\xi/C) =^{\delta}_{[B]} \mathrm{t}^g(a/C)$, keeping the induction going.

By lemma 10.17, for each $\eta \in {}^{\omega}2$, there exists some a_η satisfying $\mathrm{t}^g(a_\eta/A_n) =^{\varepsilon_n}_{[B]} \mathrm{t}^g(a_{\eta\restriction n}/A_n)$, where $\varepsilon = \sum_{i=n}^{\infty} \delta_i$. Now $|\bigcup_{n<\omega} A_n| = \aleph_0$, but if $\eta, \nu \in {}^{\omega}2$, $\eta \neq \nu$ and $n = \min\{n : \eta(n) \neq \nu(n)\}$, $\eta(n) = 0$, then

$$\begin{aligned}
&\mathbf{d}(\mathrm{t}^g(a_\eta/\bigcup_{i<\omega} A_i), \mathrm{t}^g(a_\nu/\bigcup_{i<\omega} A_i)) \\
&\geq \mathbf{d}(\mathrm{t}^g(a_\eta/A_{n+1}), \mathrm{t}^g(a_\nu/A_{n+1})) \\
&\geq \mathbf{d}(\mathrm{t}^g(a_\eta/A_{\eta\restriction n}), \mathrm{t}^g(a_\nu/A_{\eta\restriction n})) \\
&\geq \mathbf{d}(\mathrm{t}^g(a_\eta A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(a_\nu A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&\geq \mathbf{d}(\mathrm{t}^g(a_{\eta\restriction n+1} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(a_{\nu\restriction n+1} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&\quad -\mathbf{d}(\mathrm{t}^g(a_{\eta\restriction n+1} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(a_\eta A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&\quad -\mathbf{d}(\mathrm{t}^g(a_\nu A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(a_{\nu\restriction n+1} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&= \mathbf{d}(\mathrm{t}^g(a_{\eta\restriction n} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(f_{\eta\restriction n^\frown(1)}(a_{\eta\restriction n}) A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&\quad -\mathbf{d}(\mathrm{t}^g(a_{\eta\restriction n+1} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(a_\eta A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&\quad -\mathbf{d}(\mathrm{t}^g(a_\nu A_{\eta\restriction n}/A_{\eta\restriction n} \cap B), \mathrm{t}^g(a_{\nu\restriction n+1} A_{\eta\restriction n}/A_{\eta\restriction n} \cap B)) \\
&\geq \varepsilon - 2\sum_{i=n+1}^{\infty} \delta_i \\
&\geq \varepsilon - 2^{-n}\delta' \\
&\geq \varepsilon/2.
\end{aligned}$$

This contradicts ω-$\mathbf{d}$-stability and hence proves the claim. □

Lemma 10.19. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable. Then for all a, $B \subset A$, finite $C \subset A$ and $\delta > 0$ there exists a' such that $\mathrm{t}^g(a'/C) =^{\delta}_{[B]} \mathrm{t}^g(a/C)$ and $\mathrm{t}^g(a'/A)$ is isolated with respect to B.*

Proof. By lemma 10.18 we may for each $n < \omega$ choose $\delta_n, \delta'_n > 0$ a finite set $A_n \subset A$ and some a_n such that

- $A_0 = C$, $a_0 = a$, $\delta'_0 = \delta_0 = \delta$,

- $A_n \subset A_{n+1}$, A_n is finite and when $n > 0$ then δ_n, A_n $\frac{1}{n}$-isolate $\mathrm{t}^g(a_n/A)$ with respect to B,
- $\delta'_{n+1} = \frac{1}{4}\min(\delta_n, \delta'_n)$ and

$$\mathrm{t}^g(a_{n+1}/A_n) =^{\delta'_{n+1}}_{[B]} \mathrm{t}^g(a_n/A_n).$$

By lemma 10.17 there is some a' such that

$$\mathrm{t}^g(a'/A_n) =^{\varepsilon'_n}_{[B]} \mathrm{t}^g(a_n/A_n),$$

where $\varepsilon'_n = \sum_{i=n}^{\infty} \delta'_{i+1}$. We claim that $\mathrm{t}^g(a'/A)$ is isolated with respect to B. To prove this let $\varepsilon > 0$ be given. Then choose $n > 2/\varepsilon$. We show that $\delta_n/2, A_n$ ε-isolate $\mathrm{t}^g(a'/A)$. So let $f \in \mathrm{Aut}(\mathfrak{M}/A_n \cap B)$ be such that $\mathrm{d}(f(c), c) < \delta_n/2$ for all $c \in A_n$, and let $A^+ \subset A$ be finite. Since $\mathrm{t}^g(a'/A_n) =^{\varepsilon'_n}_{[B]} \mathrm{t}^g(a_n/A_n)$ and $\varepsilon'_n \leq \delta_n/2$ there is some $g \in \mathrm{Aut}(\mathfrak{M}(A_n \cap B))$ such that $\mathrm{d}(g(c), c) < \delta_n/2$ for all $c \in A_n$ and $g(a_n) = a'$. Since a_n is $\frac{1}{n}$-isolated, there is some $g' \in \mathrm{Aut}(\mathfrak{M}/A^+ \cap B)$, with $\mathrm{d}(g'(c), c) < \frac{1}{n}$ for $c \in A^+$ and $\mathrm{d}(g'(a_n), g(a_n)) < \frac{1}{n}$. Further, since $f \circ g \in \mathrm{Aut}(\mathfrak{M}/A_n \cap B)$ and $\mathrm{d}(f \circ g(c), c) \leq \delta_n$ for all $c \in A_n$, there is some $h' \in \mathrm{Aut}(\mathfrak{M}/A^+ \cap B)$ with $\mathrm{d}(h'(c), c) < \frac{1}{n}$ for all $c \in A^+$ and $\mathrm{d}(h'(a_n), f \circ g(a_n)) < \frac{1}{n}$. Let $g' = h' \circ (g')^{-1}$. Then $f' \in \mathrm{Aut}(\mathfrak{M}/A^+ \cap B)$, $\mathrm{d}(f'(c), c) < \frac{2}{n} < \varepsilon$ for all $c \in A^+$ and

$$\begin{aligned}
\mathrm{d}(f'(a'), f(a')) &= \mathrm{d}(f' \circ g(a_n), f \circ g(a_n)) \\
&\leq \mathrm{d}(f' \circ g(a_n), f' \circ g'(a_n)) + \mathrm{d}(f' \circ g'(a_n), f \circ g(a_n)) \\
&= \mathrm{d}(g(a_n), g'(a_n)) + \mathrm{d}(h'(a_n), f \circ g(a_n)) \\
&\geq \frac{1}{n} + \frac{1}{n} \\
&\geq \varepsilon.
\end{aligned}$$

□

10.2.2 $[B]$-saturation

Definition 10.20. A set $A \supset B$ is called $F^{\mathfrak{M}}_{\lambda}$-$[B]$*-saturated* if for all $A' \subset A$ with $|A'| < \lambda$, all $a \in \mathfrak{M}$ and $\varepsilon > 0$ there is $f \in \mathrm{Aut}(\mathfrak{M}/A' \cap B)$ such that $\mathrm{d}(c, f(c)) < \varepsilon$ for all $c \in A'$ and $f(a) \in A$.

Remark 10.21. Note that if $B' \subset B$ then $F^{\mathfrak{M}}_{\lambda}$-$[B]$-saturation implies $F^{\mathfrak{M}}_{\lambda}$-$[B']$-saturation.

Fact 10.22. *All complete* $F^{\mathfrak{M}}_{\omega}$-$[\emptyset]$*-saturated sets are models.*

Proof. This is proven essentially as in theorem 3.10 and corollary 3.11. The difference is that we in the construction from 3.10 instead of fixing something close to the parameters (A_n and B_n) directly require that we move them just a little. □

10.2.3 $[B]$-primary models

Definition 10.23. Let $B \subset A \subset \mathscr{C}$. $\mathscr{C}$ is *primary over* A *with respect to* B ($[B]$-primary over A for short) if

- $\mathscr{C} = \overline{C}$ where $C = A \cup \bigcup\{a_i : i < \alpha\}$,

- for all $i<\alpha$, $\mathrm{t}^g(a_i/A\cup\bigcup\{a_j : j<i\})$ is $[B]$-isolated,
- $\mathscr{C}$ is $F^{\mathfrak{M}}_{\omega}$-$[B]$-saturated.

Remark 10.24. Note that the a_i's in the definition are finite tuples and not single elements.

Remark 10.25. It is easy to see that if $B\subset A$ and A is $F^{\mathfrak{M}}_{\lambda}$-$[B]$-saturated, then $\overline{A}$ is $F^{\mathfrak{M}}_{\lambda}$-$[B]$-saturated.

Lemma 10.26. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable. Then for all A and B with $A\supset B$ there exists a $[B]$-primary model over A.*

Proof. By fact 10.22 and remark 10.25 it is enough to construct a $F^{\mathfrak{M}}_{\omega}$-$[B]$-saturated set C of the form $C=A\cup\bigcup\{a_i : i<\alpha\}$, where for all $i<\alpha$, $\mathrm{t}^g(a_i/A\cup\bigcup\{a_j : j<i\})$ is $[B]$-isolated.

We define by induction on i sets A_i and finite tuples a_i such that $A_i=A\cup\bigcup\{a_j : j<i\}$ and $\mathrm{t}^g(a_i/A_i)$ is $[B]$-isolated. If A_α has been defined and is not $F^{\mathfrak{M}}_{\omega}$-$[B]$-saturated, then there is some finite $A'\subset A_\alpha$, some $a\in\mathfrak{M}$ and $\delta>0$ such that no $f\in\mathrm{Aut}(\mathfrak{M}/A'\cap B)$ satisfies $\mathrm{d}(f(c),c)<\delta$ for all $c\in A'$ and $f(a)\in A_\alpha$. Among these choose A', a and δ such that $j_{a,A',\delta}=\min\{j : A'\subseteq A_j\}$ is minimal. Let $n>1/\delta$. Then by lemma 10.19, let a' be such that

$$\mathrm{t}^g(a'/A')=^{1/n}_{[B]}\mathrm{t}^g(a/A')$$

and $\mathrm{t}^g(a'/A_\alpha)$ is $[B]$-isolated. Then define $a_\alpha=a'$. Changing δ to some $1/n$ makes sure that we do not treat the same type more than ω times. The construction will then terminate essentially as in [She90, Theorem IV.3.1].

Then we take the metric closure of the constructed set and by 10.22 and remark 10.25 we are done. □

Definition 10.27. Assume $B\subset A\subset C$. Then C is $[B]$-*atomic over* A if for every $c\in C$, $\mathrm{t}^g(c/A)$ is $[B]$-isolated.

Lemma 10.28. *Assume $B\subset A\subset\mathscr{C}$ and $\mathscr{C}$ is $[B]$-primary over A. Then $\mathscr{C}$ is $[B]$-atomic over A.*

Proof. By definition $\mathscr{C}=\overline{C}$ for some $C=A\cup\bigcup\{a_i : i<\alpha\}$ where for each $i<\alpha$, $\mathrm{t}^g(a_i/A\cup\bigcup\{a_j : j<i\})$ is $[B]$-isolated. Hence, first prove by induction on $i<\alpha$, that for each $a\in A_i$, $\mathrm{t}^g(a/A)$ is $[B]$-isolated. This proves that C is atomic. For $a\in\mathscr{C}$, the $[B]$-$\varepsilon/3$-isolation of some $a'\in C$ with $\mathrm{d}(a',a)<\varepsilon/3$ implies the $[B]$-ε-isolation of $\mathrm{t}^g(a/A)$, completing the proof. □

Lemma 10.29. *Assume $\mathbb{K}$ is ω-$\mathbf{d}$-stable, A is countable, $A\subset\mathscr{A}$ and $\mathscr{A}$ realizes all types over finite subsets of A. Assume further that some $\delta>0$ and finite $A'\cup B'\subset A\cup B$ ε-isolate $\mathrm{t}^g(a/\mathscr{A}\cup B)$ with respect to $\mathscr{A}$ and $B'\downarrow^{\delta}_{A}\mathscr{A}\cup C$. Then $a\downarrow^{2\varepsilon}_{A}\mathscr{A}\cup C$. Moreover if $\mathscr{A}$ is $F^{\mathfrak{M}}_{\omega}$-saturated and $B\downarrow^{0}_{\mathscr{A}}C$ then for every a such that $\mathrm{t}^g(a/\mathscr{A}\cup B)$ is $[\mathscr{A}]$-isolated, $a\downarrow^{0}_{\mathscr{A}}C$.*

Proof. Let $\delta>0$ and some finite $A'\cup B'\subset A\cup B$ $[\mathscr{A}]$-ε-isolate $\mathrm{t}^g(a/\mathscr{A}\cup B)$ and let some finite $A''\subset A$ witness $B'\downarrow^{\delta}_{A}C$. Let $E=A'\cup A''$. We claim that $\mathrm{t}^g(a/\mathscr{A}\cup C)$ does not 2ε-split over E. So let $b,c\in\mathscr{A}\cup C$ and $g\in\mathrm{Aut}(\mathfrak{M}/E)$ such that $g(c)=b$.

Case 1: $b \in \mathscr{A}$. Since $B' \downarrow^{\delta}_{E} \mathscr{A} \cup C$,

$$\mathbf{d}(\mathrm{t}^g(B'/E \cup b), \mathrm{t}^g(g(B')/E \cup b)) < \delta.$$

Hence there is $g' \in \mathrm{Aut}(\mathfrak{M}/E \cup b)$ with $\mathbf{d}(g' \circ g(B'), B') < \delta$. Now $g' \circ g \in \mathrm{Aut}(\mathfrak{M}/E)$ and $\mathrm{d}(g' \circ g(B'), B') < \delta$. Since δ and $A' \cup B' \subset E \cup B'$ $[\mathscr{A}]$-ε-isolate $\mathrm{t}^g(a/\mathscr{A} \cup B)$, for every finite $A^+ \subset \mathscr{A}$ we have

$$\mathbf{d}(\mathrm{t}^g(g' \circ g(a)/A^+), \mathrm{t}^g(a/A^+)) < \varepsilon,$$

especially since $\mathrm{t}^g(g' \circ g(a)/E \cup b) = \mathrm{t}^g(g(a)/E \cup b)$,

$$\mathbf{d}(\mathrm{t}^g(g(a)/E \cup b), \mathrm{t}^g(a/E \cup b)) < \varepsilon,$$

which proves non-ε-splitting if $b \in \mathscr{A}$.

Case 2: $b \notin \mathscr{A}$. By the saturation properties of $\mathscr{A}$ let $g' \in \mathrm{Aut}(\mathfrak{M}/E)$, $g'(b) \in \mathscr{A}$. Then we use case 1 and deduce:

$$\begin{aligned}
&\mathbf{d}(\mathrm{t}^g(g(a)/E \cup b), \mathrm{t}^g(a/E \cup b)) \\
&= \mathbf{d}(\mathrm{t}^g(g' \circ g(a)/E \cup g'(b)), \mathrm{t}^g(g'(a)/E \cup g'(b))) \\
&\leq \mathbf{d}(\mathrm{t}^g(g' \circ g(a)/E \cup g'(b)), \mathrm{t}^g(a/E \cup g'(b))) \\
&\quad + \mathbf{d}(\mathrm{t}^g(a/E \cup g'(b)), \mathrm{t}^g(g'(a)/E \cup g'(b))) \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Hence in either case $a \downarrow^{2\varepsilon}_{A} \mathscr{A} \cup C$. The moreover part clearly follows. □

10.3 Main theorem

Theorem 10.30. *Assume $\mathbb{K}$ is κ-categorical for some uncountable κ and metricly homogeneous. Assume further that* either $\kappa \geq \aleph_2$ or *there are $F^{\mathfrak{M}}_{\omega}$-saturated separable models. Then every $F^{\mathfrak{M}}_{\aleph_1}$-saturated model $\mathscr{A}$ is saturated.*

Proof. Let $|\mathscr{A}| = \lambda$. We wish to show that $\mathscr{A}$ is $F^{\mathfrak{M}}_{\lambda}$-$\mathbf{d}$-saturated, since then the claim follows using theorem 3.7. So let $B \subset \mathscr{A}$, $|B| < \lambda$, $q \in S(B)$ and $\varepsilon > 0$.

Now note that by κ-categoricity, $\mathbb{K}$ is ω-$\mathbf{d}$-stable (Corollary 5.8). Further by metric homogeneity and theorem 10.9, $\mathbb{K}$ is ξ-$\mathbf{d}$-stable for every $\xi \geq \aleph_0$ and by theorem 10.10 there exists saturated models of density ξ for each uncountable ξ. Hence the unique model of density κ is saturated, and all larger models are at least $F^{\mathfrak{M}}_{\kappa}$-saturated.

We may assume λ is regular since once we have proven the theorem for regular cardinals, the claim will follow for singular cardinalities by taking an $F^{\mathfrak{M}}_{\aleph_1}$-saturated submodel of regular density containing the parameters.

Since $\mathscr{A}$ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated, we may assume B is. Since $\mathbb{K}$ is ω-$\mathbf{d}$-stable, $F^{\mathfrak{M}}_{\aleph_1}$-saturation implies strong $F^{\mathfrak{M}}_{\aleph_1}$-saturation (lemma 6.3). So we may choose models $\mathscr{B}_i$ and elements b_i, for $i < \lambda$, such that

- $\mathscr{B}_0 = B$,
- $\mathscr{B}_i \subset \mathscr{A}$,
- $|\mathscr{B}_i| < \lambda$

- $\mathscr{B}_i$ is (strongly) $F^{\mathfrak{M}}_{\aleph_1}$-saturated,
- $b_i \in \mathscr{A} \backslash \mathscr{B}_i$,
- $\mathscr{B}_{i+1} \supseteq \mathscr{B}_i \cup b_i$
- for limit i, $\mathscr{B}_i \supseteq \bigcup_{j<i} \mathscr{B}_j$ and for $cf(i) \geq \aleph_1$ we have equality.

By lemma 4.11 for each $i < \lambda$ choose a countable set $A_i \subset \mathscr{B}_i$ such that $b_i \downarrow^0_{A_i} \mathscr{B}_i$. Let $f(i)$ be the least j for which $A_i \subset \mathscr{B}_j$. then for $cf(i) \geq \aleph_1$, $f(i) < i$ so by Fodor's lemma there is a stationary set $S \subset \lambda$ and $i^* < \lambda$ such that $A_i \subset \mathscr{B}_{i^*}$ for all $i \in S$. Since S has size λ we may reenumerate $\mathscr{B}_i$ for $i \in S$, starting from $\mathscr{B}_0 = \mathscr{B}_{i^*}$. Now we have an increasing chain of $F^{\mathfrak{M}}_{\aleph_1}$-saturated models such that

- $q \in S(\mathscr{B}_0)$ (we just take any extension of the original q),
- for every $i < \lambda$, $b_i \downarrow^0_{\mathscr{B}_0} \mathscr{B}_i$.

Since $|\mathscr{B}_0| < \lambda$ and $\mathbb{K}$ is $|\mathscr{B}_0|$-$\mathbf{d}$-stable, the set $\{\mathrm{t}^g(b_i/\mathscr{B}_0) : i < \lambda\}$ has an accumulation point in $S(\mathscr{B}_0)$, denote it (one of them) by p. Now for each $k < \omega$ let $(i^k_n)_{n<\omega}$ be an increasing sequence of indices $< \lambda$ such that $\mathbf{d}(\mathrm{t}^g(b_{i^k_n}/\mathscr{B}_0), p) < 1/k$.

Next let $A \subset \mathscr{B}_0$ be a countable set such that the types p, q and $\mathrm{t}^g(b_{i^k_n}/\mathscr{B}_0)$, for $k, n < \omega$, are 0-independent over A. Let $\mathscr{A}^+ \subset \mathscr{B}_0$ be $F^{\mathfrak{M}}_{\omega}$-saturated and such that $A \subset \mathscr{A}^+$.

We may construct a "0-Morley sequence" for p over A as follows: Let a_0 realize p. Then $a_0 \downarrow^0_A \mathscr{B}_0$. When a_i has been defined for all $i < \alpha$, let $D_\alpha \supset \mathscr{B}_0 \cup \{a_i : i < \alpha\}$ be $F^{\mathfrak{M}}_{\aleph_1}$-saturated and by 4.15 let a_α realize $p = \mathrm{t}^g(a_0/\mathscr{B}_0)$ and satisfy $a_\alpha \downarrow^0_A D_\alpha$. This way construct a sequence I of length $> \kappa$ such that $I \downarrow^0_A \mathscr{B}_0$. Let $\mathscr{M}$ be $[\mathscr{A}^+]$-primary over $\mathscr{A}^+ \cup I$. Then $|\mathscr{M}| > \kappa$ so $\mathscr{M}$ is $F^{\mathfrak{M}}_{\kappa}$-saturated. Now either $\kappa \geq \aleph_2$ or we can assume $\mathscr{A}^+$ to be separable, so $\mathscr{M}$ realizes $q \restriction \mathscr{A}^+$, by say a. By lemma 10.28, $\mathscr{M}$ is $[\mathscr{A}^+]$-atomic over $\mathscr{A}^+ \cup I$, so by lemma 10.29 $a \downarrow^0_{\mathscr{A}^+} \mathscr{B}_0$. Then by lemma 4.12 a realizes q and $a \downarrow^0_A \mathscr{B}_0$.

Now let $\delta > 0$ and some finite $D \cup J \subset \mathscr{A}^+ \cup I$ $f^{MH}(\varepsilon)/2$-isolate $\mathrm{t}^g(a/\mathscr{A}^+ \cup I)$ with respect to $\mathscr{A}^+$. Then let $\mathscr{A}'$ be a separable model $\supset A \cup D$ realizing all types over finite subsets of $A \cup D$. We wish to move $J = \{j_n : n < l\}$ into $\mathscr{A}$ close enough to some b_i's to be $\delta/2$-independent. So let $k > 1/f^{MH}(\delta/2)$ and note that since $\mathbf{d}(\mathrm{t}^g(b_{\frac{k}{n}}/\mathscr{B}_0), p) < 1/k$ for each $n < \omega$, there is some $j^+_0 (\in \mathfrak{M})$ realizing p with $\mathrm{d}(j^+_0, b_{i^k_0}) < 1/k < f^{MH}(\delta/2)$. By $F^{\mathfrak{M}}_{\aleph_1}$-saturation of $\mathscr{B}_{i^k_0+1}$, we may realize $\mathrm{t}^g(j^+_0/\mathscr{A}' \cup b_{i^k_0})$ in $\mathscr{B}_{i^k_0+1}$ by some j'_0. Then $\mathrm{t}^g(j'_0/\mathscr{A}') = \mathrm{t}^g(j^+_0/\mathscr{A}') = \mathrm{t}^g(j_0/\mathscr{A}')$ and $\mathrm{d}(j'_0, b_{i^k_0}) = \mathrm{d}(j^+_0, b_{i^k_0}) < f^{MH}(\delta/2) \leq \delta/2$ and there is some $f_0 \in \mathrm{Aut}(\mathfrak{M}/\mathscr{A}')$ mapping j_0 to j'_0. Next assume $n < l$ and we have defined elements j'_m, for $m \leq n$ and an automorphism f_n such that

- $j'_m = f_n(j_m) \in \mathscr{B}_{i^k_m+1}$,
- $f_n \in \mathrm{Aut}(\mathfrak{M}/\mathscr{A}')$,
- $\mathrm{d}(j'_m, b_{i^k_m}) < \delta/2$.

Now

$$\begin{aligned}\mathbf{d}(\mathrm{t}^g(f_n(j_{n+1})/\mathscr{A}'), \mathrm{t}^g(b_{i^k_{n+1}}/\mathscr{A}')) &= \mathbf{d}(\mathrm{t}^g(j_{n+1}/\mathscr{A}'), \mathrm{t}^g(b_{i^k_{n+1}}/\mathscr{A}')) \\ &= \mathbf{d}(p \restriction \mathscr{A}', \mathrm{t}^g(b_{i^k_{n+1}}/\mathscr{A}')) \\ &< 1/k < f^{MH}(\delta/2).\end{aligned}$$

Further since $j_{n+1} \downarrow^0_A \mathscr{B}_0 \cup \bigcup\{j_m : m \leq n\}$, we have

$$f_n(j_{n+1}) \downarrow^0_A \mathscr{A}' \cup \bigcup\{j'_m : m \leq n\}.$$

Also, since $j'_m \in \mathscr{B}_{i^k_n+1} \subseteq \mathscr{B}_{i^k_{n+1}}$ for $m \leq n$, we have

$$b_{i^k_{n+1}} \downarrow^0_A \mathscr{A}' \cup \bigcup\{j'_m : m \leq n\}.$$

So by lemma 10.4

$$\mathbf{d}(\mathrm{t}^g(f_n(j_{n+1})/\mathscr{A}' \cup \bigcup\{j'_m : m \leq n\}), \mathrm{t}^g(b_{i^k_{n+1}}/\mathscr{A}' \cup \bigcup\{j'_m : m \leq n\})) < \delta/2.$$

So by the same technique as for finding j'_0 we can find $j'_{n+1} \in \mathscr{B}_{i^k_{n+1}+1}$ with $\mathrm{d}(j'_{n+1}, b_{i^k_{n+1}}) < \delta/2$ and some $g \in \mathrm{Aut}(\mathfrak{M}/\mathscr{A}' \cup \bigcup\{j'_m : m \leq n\})$ moving $f_n(j_{n+1})$ to j'_{n+1}. Then we let $f_{n+1} = g \circ f_n$ and continue the induction. Finally we let

$$f^+ = \bigcup_{n<l} f_n \restriction (\mathscr{A}' \cup \bigcup\{j_m : m \leq n\})$$

which is well-defined and type-preserving and hence extends to an automorphism $f' \in \mathrm{Aut}(\mathfrak{M}/\mathscr{A}')$. By further realizing $\mathrm{t}^g(f'(a)/\mathscr{A}' \cup \bigcup\{j'_n : n < l\})$ in $\mathscr{A}$ we find $f \in \mathrm{Aut}(\mathfrak{M}/\mathscr{A}')$ moving both J and a into $\mathscr{A}$.

Denote $J' = \{j'_n : n < l\}$ and $a' = f(a)$. Now $\mathrm{d}(J', (b_{i^k_n})_{n<l}) < \delta/2$ and $(b_{i^k_n})_{n<l} \downarrow^0_A \mathscr{B}_0$ so by lemma 10.5 $J' \downarrow^\delta_A \mathscr{B}_0$. Further since δ and $D \cup J$ $[\mathscr{A}^+]$-$f^{MH}(\varepsilon)/2$-isolate $\mathrm{t}^g(a/\mathscr{A}^+ \cup I)$, δ and $D \cup J'$ $[\mathscr{A}']$-$f^{MH}(\varepsilon)/2$-isolate $\mathrm{t}^g(a'/\mathscr{A}' \cup f(I))$. So by lemma 10.29 $a' \downarrow^{f^{MH}(\varepsilon)}_A \mathscr{B}_0$.

Finally we remember that $a \downarrow^0_A \mathscr{B}_0$ and $\mathrm{t}^g(a'/\mathscr{A}') = \mathrm{t}^g(a/\mathscr{A}')$. Hence lemma 10.4 implies

$$\mathbf{d}(\mathrm{t}^g(a'/\mathscr{B}_0), \mathrm{t}^g(a/\mathscr{B}_0)) \leq \varepsilon$$

proving $F^{\mathfrak{M}}_\lambda$-$\mathbf{d}$-saturation. □

Corollary 10.31. *If $\mathbb{K}$ is metricly homogeneous and κ-categorical for some uncountable κ and either $\kappa > \aleph_1$ or separable $F^{\mathfrak{M}}_\omega$-saturated models exist, then there exists some $\xi < \beth_{\mathfrak{c}^+}$ such that $\mathbb{K}$ is categorical in all $\lambda \geq \min\{\kappa, \xi\}$.*

Proof. Note that lemma 8.2 uses the assumption of $\kappa^{\aleph_0} = \kappa$ only to get stability. However with metric homogeneity this assumption becomes unnecessary. Hence the proof of the lemma shows that for some $\xi < \beth_{\mathfrak{c}^+}$, every model of *cardinality* at least ξ is $F^{\mathfrak{M}}_{\aleph_1}$-saturated. Hence of course all models of *density*$\geq \xi$ are $F^{\mathfrak{M}}_{\aleph_1}$-saturated, and the claim follows by theorem 10.30. □

Remark 10.32. Ben-Yaacov and Usvyatsov define *d-finiteness* in [BYU07]. In our context a type $\mathrm{t}^g(a/\emptyset)$ is d-finite if for every b and $\varepsilon > 0$ there exists some $\delta > 0$ such that whenever $\mathrm{t}^g(a'/\emptyset) = \mathrm{t}^g(a/\emptyset)$ and $\mathrm{d}(a, a') \leq \delta$, there is b' such that $\mathrm{d}(b, b') \leq \varepsilon$ and $\mathrm{t}^g(a'b'/\emptyset) = \mathrm{t}^g(ab/\emptyset)$. It is easy to see that if all (finite) types are d-finite, then approximately $F^{\mathfrak{M}}_\omega$-saturated models are $F^{\mathfrak{M}}_\omega$-saturated, giving us separable $F^{\mathfrak{M}}_\omega$saturated models (and hence categoricity transfer also assuming $\kappa = \aleph_1$).

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