Finite Density on the Lattice:
problems and (some) solutions

Tobias Rindlisbacher [1]

[1] University of Helsinki, Helsinki, Finland

Saariselkä, April 7, 2018

1. Motivation: why lattice field theory at finite density?
2. Lattice QCD at finite quark density, sign problem.
3. Isospin QCD at finite density.
4. Lattice $\mathrm{SU}(2)$ principal chiral model at finite density.

Why lattice field theory at finite density?

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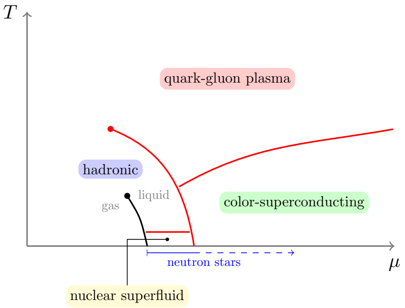
→ Conjectured QCD phase diagram:



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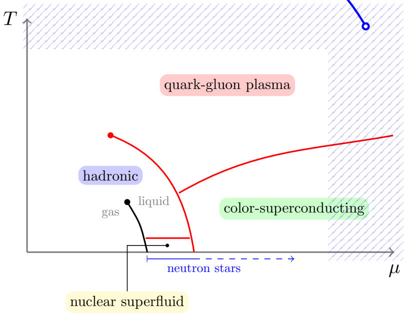
Perturbation theory (high dens./temp. → asympt. freedom)



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Why lattice field theory at finite density?

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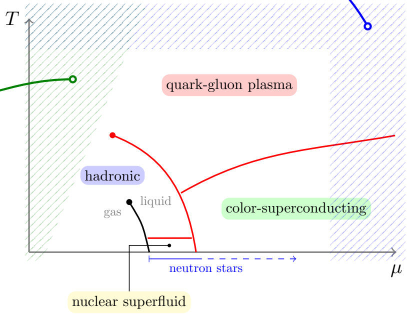
Lattice QCD & heavy ion col. exp.



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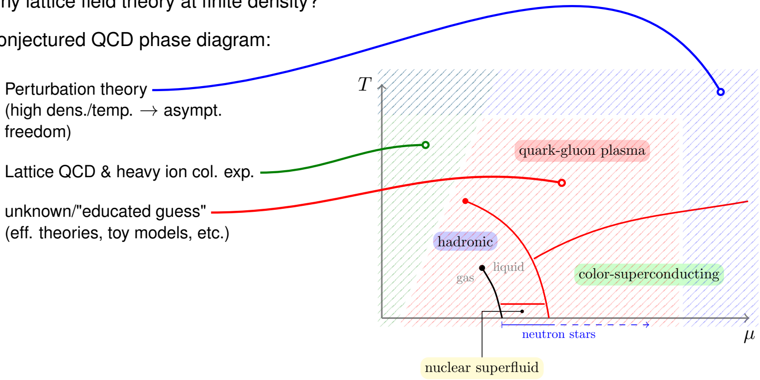

Why lattice field theory at finite density?

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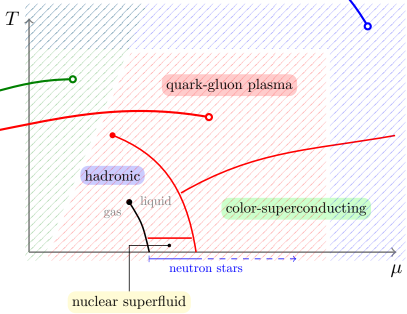
unknown/"educated guess" (eff. theories, toy models, etc.)



→ Need first-principles method to verify conjectures!

→ Non-perturbative phenomena!

- Why lattice QCD fails at finite chemical potential:

$$Z_{QCD} = \int \mathcal{D}[U,\psi,\bar{\psi}]\,\exp\big(-S_F[\psi,\bar{\psi},U] - S_G[U]\big) = \int \mathcal{D}[U]\,\mathrm{Det}(M[U])\,\mathrm{e}^{-S_G[U]}$$

Fermions need to be integrated out analytically $\rightarrow$ fermion determinant

$\rightarrow$ Monte Carlo: **integrand (incl. measure) needs to be real and non-negative!**

$\rightarrow$ It holds: $\gamma_5\,M_{x,y}(\mu)\,\gamma_5 = M^{\dagger}_{y,x}(-\mu^*) \quad\Rightarrow\quad \mathrm{Det}(M(\mu)) = \mathrm{Det}(M(-\mu^*))^*$.

- If $\mathrm{Re}(\mu) = 0$:

  $\rightarrow$ $\mathrm{Det}(M(\mu)) \in \mathbb{R}$,

  $\rightarrow$ $\mathrm{Det}(M(\mu))^2 \geq 0$ $\Rightarrow$ even numbers of mass-degenerate flavors can be simulated!

- If $\mathrm{Re}(\mu) \neq 0$:

  $\rightarrow$ $\mathrm{Det}(M(\mu)) \in \mathbb{C}$ $\Rightarrow$ "sign problem"!

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- Partition function for full QCD:

$$Z_{QCD} = \int \mathcal{D}[U]\,\mathrm{Det}(M[U])\,\mathrm{e}^{-S_G[U]} = \int \mathcal{D}[U]\,\underbrace{\frac{Det M[U]}{|\mathrm{Det}(M[U])|}}_{\mathrm{R}[U]\in\mathrm{U}(1)}\,|\mathrm{Det}(M[U])|\,\mathrm{e}^{-S_G[U]}$$

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- Partition function for full QCD:

$$Z_{QCD} = \int \mathcal{D}[U]\,\mathrm{Det}(M[\,U\,])\,\mathrm{e}^{-S_G[\,U\,]} = Z_{|QCD|} \cdot \langle \mathrm{R} \rangle_{|QCD|}$$

with:

  - partition function for "phase quenched" theory:

$$Z_{|QCD|} = \int \mathcal{D}[U]\,|\mathrm{Det}(M[\,U\,])|\,\mathrm{e}^{-S_G[\,U\,]}$$

  - expectation value of observable $\mathcal{O}$ in "phase quenched" QCD:

$$\langle \mathcal{O} \rangle_{|QCD|} = \frac{1}{Z_{|QCD|}} \int \mathcal{D}[U]\,\mathcal{O}[\,U\,]\,|\mathrm{Det}(M[\,U\,])|\,\mathrm{e}^{-S_G[\,U\,]}$$

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→ Problem: $R[U]$ highly oscillatory

$$\langle \mathrm{R} \rangle_{|QCD|} = \frac{Z_{QCD}}{Z_{|QCD|}} = \mathrm{e}^{-L^3 N_t \Delta f} \quad , \quad \Delta f = f_{QCD} - f_{|QCD|} \, .$$

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1. Lattice QCD: sign problem at finite μ

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→ Monte Carlo error goes like: err. $\propto \frac{1}{\sqrt{\#\text{ meas.}}}$

⇒ required statistics for equal accuracy $\propto e^{2 L^3 N_t \Delta f}$

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- Why is $\Delta f \neq 0$? Consider two-flavor QCD with deg. quark masses.

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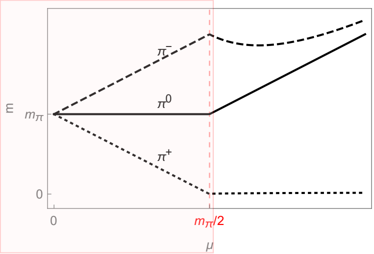
Isospin system:

$\rightarrow$ effective pion masses: $m = m_\pi - 2\mu Q$, $Q = 0, \pm 1$, (for sufficiently small μ)



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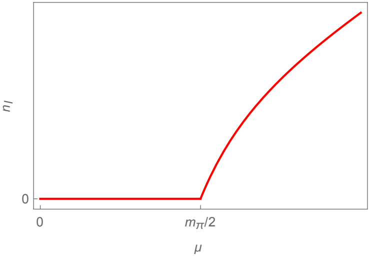

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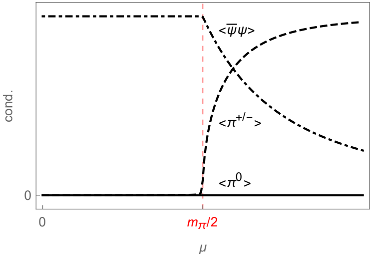

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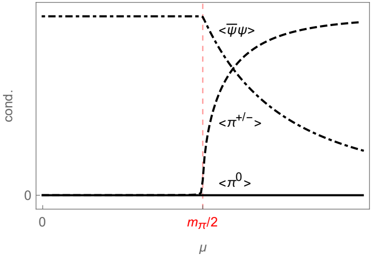

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→ as long $\mu < m_\pi/2$ so that vacuum at $\langle\bar{\psi}\psi\rangle \neq 0$, $\langle\pi^\pm\rangle = 0$:

can (approx.) identify: QCD: (u,d), $(\bar{u},\bar{d})$ ↔ $|QCD|$: $(u,\bar{d})$, $(\bar{u},d)$ ⇒ Δf small .

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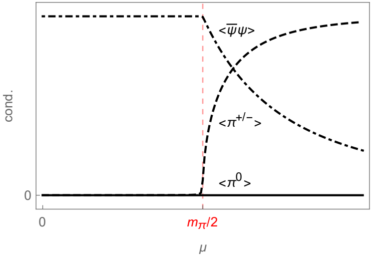

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$\rightarrow$ identification breaks down when $\langle\pi^\pm\rangle \neq 0 \Rightarrow \boxed{\Delta f \text{ large}}$!

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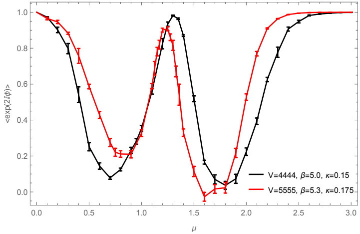
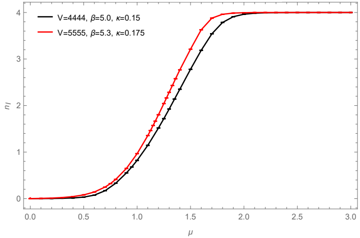
→ Behaviour of $\langle \mathrm{R} \rangle_{|QCD|}(\mu)$:

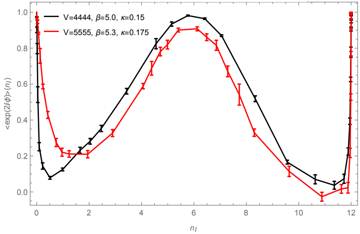
average sign $\langle R \rangle_{|QCD|}$ vs. μ



n_I vs. μ



average sign $\langle R \rangle_{|QCD|}$ vs. n_I



→ Average sign drops dramatically as soon as n_I becomes non-zero.

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- Attempts to overcome sign problem in finite density lattice QCD?

→ complexification and deformation of path integral's domain of integration:

→ hol. gradient flow (with/without neural network), Basar's talk, or [Alexandru et al., 2017] ,

→ path optimization (with/without neural network), [Mori et al., 2017]

→ cluster decomposition, [Wenger et al., 2017]

→ strong coupling and/or (spatial) hopping expansion, [Forcrand et al.],[Philipsen et al.]

→ ...

→ spatial hopping/loop expansion + mean field.

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- Circumventing the finite density sign problem in QCD:

  Single flavor LQCD partition function,

$$Z = \int \mathcal{D}[U] \, \mathrm{Det}(D[U]) \, \mathrm{e}^{-S_g[U]} \, ,$$

  with Wilson's lattice gauge action,

$$S_g[U] = \frac{\beta}{3} \sum_x \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{Tr}\left(\mathbb{1} - U_{\mu\nu}(x)\right) , \quad U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\widehat{\mu}) U_\mu^\dagger(x+\widehat{\nu}) U_\nu^\dagger(x) \, ,$$

  and Wilson's lattice Dirac operator,

$$\begin{aligned} D_{xIa,yJb}[U] = {} & \delta_{xy}\delta_{IJ}\delta_{ab} \\ & - \kappa \underbrace{\sum_{\nu=1}^{3} \left(\delta_{x+\widehat{\nu},y} (\mathbb{1} - \gamma_\nu)_{ab} U_{\nu,IJ}(x) + \delta_{x-\widehat{\nu},y} (\mathbb{1} + \gamma_\nu)_{ab} U^\dagger_{\nu,IJ}(x-\widehat{\nu}) \right)}_{S_{xIa,yJb}} \\ & - \kappa \underbrace{\left(\delta_{x+\widehat{4},y} (\mathbb{1} - \gamma_4)_{ab} U_{4,IJ}(x) \, \mathrm{e}^{\mu} + \delta_{x-\widehat{4},y} (\mathbb{1} + \gamma_4)_{ab} U^\dagger_{4,IJ}(x-\widehat{4}) \, \mathrm{e}^{-\mu} \right)}_{T_{xIa,yJb}} \, . \end{aligned}$$

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- Rewrite single flavor partition function:

$$Z = \int \mathcal{D}[U] \, \mathrm{Det}(D[U]) \, \mathrm{e}^{-S_g[U]}$$

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1. Lattice QCD: effective model

[T. Rindlisbacher & P. de Forcrand, JHEP 51 (2016)]

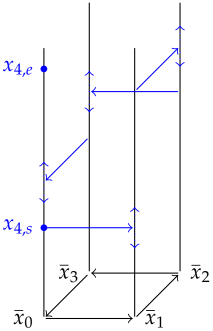
- Rewrite single flavor partition function:

$$Z = \int \mathcal{D}[U]\,\mathrm{Det}(\mathbb{1} - \kappa\, T) \underbrace{\left(\prod_{s_0} \prod_{\{C_{s_0}\}} \det_{c,d,t}\left(\mathbb{1} - \kappa_s^{s_0} \tilde{M}_{C_{s_0}}\right)\right)}_{\text{"spatial loop" expansion of } \mathrm{Det}(D[U])} \underbrace{\prod_p \left\{ 1 + \sum_{r \neq 0} d_r\, a_r(\beta)\, \chi_r(U_p) \right\}}_{\text{character expansion of } \mathrm{e}^{-S_g[U]}}$$

with

$$\tilde{M}_{C_{s_0}} = \prod_{\substack{i=0 \\ \bar{x}_i \in C_{s_0}}}^{s_0 - 1} S_{\bar{x}_i, \bar{x}_{((i+1) \bmod s_0)}} \left(\mathbb{1} - \kappa\, T\right)^{-1}_{\bar{x}_{((i+1) \bmod s_0)}} .$$

$$\left(\tilde{M}_{C_4}\right)_{a,I,x_{4,s};b,J,x_{4,e}}$$



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- Effective nearest-neighbor Polyakov loop model obtained by truncating:
  - $s_0 \leq 2$ (loops spanned by no more than two neighboring spatial sites)
  - $p \in P_t$ (P_t: set of time-like plaquettes only)
  - $r = f$ (fundamental plaquettes only)

$$\rightarrow Z \approx \int \mathcal{D}[U]\, \mathrm{Det}(\mathbb{1} - \kappa T) \left(\prod_{\langle \bar{x}, \bar{y} \rangle} \det_{c,d,t}\left(\mathbb{1} - \kappa_s^2 \tilde{M}_{\bar{x},\bar{y}}\right) \right) \prod_{p \in P_t} \left\{ 1 + 3\, a_f(\beta)\, \mathrm{tr}_c(U_p) \right\}$$

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- Take terms up to order $\mathcal{O}\left(\kappa_s^2\, a_f^{n_t(\beta)}\right)$ and integrate out spatial links:

$$Z_{eff} = \int \mathcal{D}[P]\left(\prod_{\bar{x}} \mathrm{det}_c^2\left(\mathbb{1} + (2\kappa\, \mathrm{e}^{\mu})^{n_t} P_{\bar{x}}\right) \mathrm{det}_c^2\left(\mathbb{1} + \left(2\kappa\, \mathrm{e}^{-\mu}\right)^{n_t} P_{\bar{x}}^{\dagger}\right)\right) \mathrm{e}^{-S_{eff}(n_t,\kappa,\beta,\mu)[P,P^{\dagger}]}$$

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$\rightarrow$ can be expressed entirely in terms of $L_x = \mathrm{tr}_c(P_x)$ and $\bar{L}_x = \mathrm{tr}_c(P_x^{\dagger})$.

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- Full set of nearest neighbor hoppings:

$$\begin{aligned}\det_{c,d,t}\left(\mathbb{1}-\kappa_s^2\tilde{M}_{\bar{x},\bar{y}}\right) &= 1-\kappa_s^2\,\mathrm{tr}_{c,d,t}\left(\tilde{M}_{\bar{x},\bar{y}}\right)\\ &+\frac{\kappa_s^4}{2}\left(\mathrm{tr}^2_{c,d,t}\left(\tilde{M}_{\bar{x},\bar{y}}\right)-\mathrm{tr}_{c,d,t}\left(\tilde{M}^2_{\bar{x},\bar{y}}\right)\right)\\ &-\frac{\kappa_s^6}{6}\left(\mathrm{tr}^3_{c,d,t}\left(\tilde{M}_{\bar{x},\bar{y}}\right)-3\,\mathrm{tr}_{c,d,t}\left(\tilde{M}_{\bar{x},\bar{y}}\right)\mathrm{tr}_{c,d,t}\left(\tilde{M}^2_{\bar{x},\bar{y}}\right)+2\,\mathrm{tr}_{c,d,t}\left(\tilde{M}^3_{\bar{x},\bar{y}}\right)\right)\\ &+\ldots\\ &=\sum_{k=0}^{12\,n_t}\kappa_s^{2k}\,c_{12\,n_t-k}\left(\tilde{M}_{\bar{x},\bar{y}}\right),\end{aligned}$$

where for a $n\times n$ matrix A, the $c_k(A)$ are defined by,

$$\chi_A(\lambda)=\det(\mathbb{1}\lambda-A)=\sum_{k=0}^{n}\lambda^k\,c_k(A).$$

- For simplicity, start with leading terms up to quadratic order in κ_s and require

$$Z_{eff}=\int\mathcal{D}[P]\left(\prod_{\bar{x}}\det_c^2\left(\mathbb{1}+(2\kappa\,\mathrm{e}^{\mu})^{n_t}P_{\bar{x}}\right)\det_c^2\left(\mathbb{1}+\left(2\kappa\,\mathrm{e}^{-\mu}\right)^{n_t}P^{\dagger}_{\bar{x}}\right)\right)\mathrm{e}^{-S_{f,eff}(n_t,\kappa,\beta,\mu)-S_{g,eff}(n_t,\beta)}$$

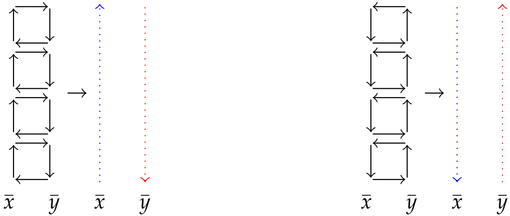
to coincide with Z up to order $\mathcal{O}\left(\kappa_s^2\,a_f^{n_t}(\beta)\right) \quad\Longrightarrow\quad S_{f,eff}(n_t,\kappa,\beta,\mu)$ and $S_{g,eff}(n_t,\beta)$

- Effective gauge action:

$\longrightarrow$ Integrate out spatial links in $\mathcal{O}\left(\kappa_s^0\right)$ piece of Z:

$$\int \mathcal{D}[U_s] \prod_{p \in P_t} \left\{ 1 + 3\, a_f(\beta)\, \mathrm{tr}_c(U_p) \right\}$$

$$= \prod_{\langle \bar{x}, \bar{y} \rangle} \left\{ 1 + a_f^{n_t}(\beta) \left(\mathrm{tr}_c(P_{\bar{x}})\, \mathrm{tr}_c(P_{\bar{y}}^\dagger) + \mathrm{tr}_c(P_{\bar{x}}^\dagger)\, \mathrm{tr}_c(P_{\bar{y}}) \right) \right\},$$



$$\Longrightarrow \quad -S_{g,eff}(n_t, \beta) = a_f^{n_t}(\beta) \sum_{\langle \bar{x}, \bar{y} \rangle} \left(\mathrm{tr}_c(P_{\bar{x}})\, \mathrm{tr}_c(P_{\bar{y}}^\dagger) + \mathrm{tr}_c(P_{\bar{x}}^\dagger)\, \mathrm{tr}_c(P_{\bar{y}}) \right),$$

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- Effective fermion action (interaction part):

$\longrightarrow$ Integrate out spatial links in $\mathcal{O}\left(\kappa_s^2\right)$ - $\mathcal{O}\left(\kappa_s^2\, a_f^{n_t}(\beta)\right)$ pieces of Z:

$$-\kappa_s^2 \int \mathcal{D}[U_s] \sum_{\langle\bar{x},\bar{y}\rangle} \mathrm{tr}_{c,d,t}\left(S_{\bar{x},\bar{y}}\,(\mathbb{1}-\kappa\,T)_{\bar{y}}^{-1}\,S_{\bar{y},\bar{x}}\,(\mathbb{1}-\kappa\,T)_{\bar{x}}^{-1}\right) \prod_{p\in P_t}\left(1+3\,a_f(\beta)\,\mathrm{tr}_c(U_p)\right)$$

$$\Longrightarrow \quad -S_{f,eff}(n_t,\kappa,\beta,\mu) = -\underbrace{S_{f,eff,0}(n_t,\kappa,\mu)}_{\mathcal{O}\left(\kappa_s^2\, a_f^0(\beta)\right)} - \underbrace{S_{f,eff,1}(n_t,\kappa,\beta,\mu)}_{\mathcal{O}\left(\kappa_s^2\, a_f^1(\beta)\right)-\mathcal{O}\left(\kappa_s^2\, a_f^{n_t-1}(\beta)\right)} - \underbrace{S_{f,eff,2}(n_t,\kappa,\beta,\mu)}_{\mathcal{O}\left(\kappa_s^2\, a_f^{n_t}(\beta)\right)},$$

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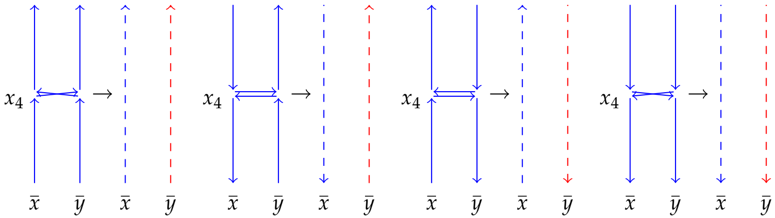
- For example the contribution at order $\mathcal{O}\left(\kappa_s^2\, a_f^0(\beta)\right)$:



$$\Longrightarrow\; -S_{f,eff,0}(n_t,\kappa,\mu) = -\frac{2\kappa^2 n_t}{3}$$
$$\cdot \sum_{\langle\bar{x},\bar{y}\rangle}\left(\mathrm{tr}_c\Big(\big(\mathbb{1}+(2\kappa\,\mathrm{e}^{\mu})^{n_t}P_{\bar{x}}\big)^{-1}\Big)-\mathrm{tr}_c\Big(\big(\mathbb{1}+\left(2\kappa\,\mathrm{e}^{-\mu}\right)^{n_t}P^{\dagger}_{\bar{x}}\big)^{-1}\Big)\right)$$
$$\cdot\left(\mathrm{tr}_c\Big(\big(\mathbb{1}+(2\kappa\,\mathrm{e}^{\mu})^{n_t}P_{\bar{y}}\big)^{-1}\Big)-\mathrm{tr}_c\Big(\big(\mathbb{1}+\left(2\kappa\,\mathrm{e}^{-\mu}\right)^{n_t}P^{\dagger}_{\bar{y}}\big)^{-1}\Big)\right),$$

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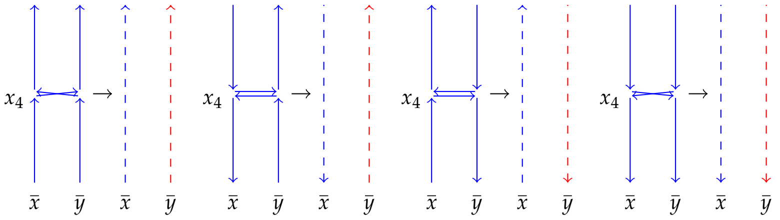

$$\Longrightarrow -S_{f,eff,0}(n_t,\kappa,\mu) = -\frac{2\kappa^2 n_t}{3}$$
$$\cdot \sum_{\langle \bar{x},\bar{y}\rangle} \left(\mathrm{tr}_c\left(\left(\mathbb{1}+(2\kappa\,\mathrm{e}^{\mu})^{n_t} P_{\bar{x}}\right)^{-1}\right) - \mathrm{tr}_c\left(\left(\mathbb{1}+\left(2\kappa\,\mathrm{e}^{-\mu}\right)^{n_t} P_{\bar{x}}^{\dagger}\right)^{-1}\right)\right)$$
$$\cdot \left(\mathrm{tr}_c\left(\left(\mathbb{1}+(2\kappa\,\mathrm{e}^{\mu})^{n_t} P_{\bar{y}}\right)^{-1}\right) - \mathrm{tr}_c\left(\left(\mathbb{1}+\left(2\kappa\,\mathrm{e}^{-\mu}\right)^{n_t} P_{\bar{y}}^{\dagger}\right)^{-1}\right)\right),$$

- Higher orders ...

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- Partition function for effective model:

$$Z_{eff} = \int \mathcal{D}[P] \left(\prod_{\bar{x}} \det_c^2 \left(\mathbb{1} + (2\kappa\, e^{\mu})^{n_t} P_{\bar{x}} \right) \det_c^2 \left(\mathbb{1} + \left(2\kappa\, e^{-\mu}\right)^{n_t} P_{\bar{x}}^{\dagger} \right) \right) e^{-S_{eff,tot}(n_t,\kappa,\beta,\mu)}$$

where

$$-S_{eff,tot}(n_t,\kappa,\beta,\mu) = -\underbrace{S_{g,eff}(n_t,\beta)}_{\mathcal{O}\left(\kappa_s^0\, a_f^{n_t}(\beta)\right)} - \underbrace{S_{f,eff,0}(n_t,\kappa,\mu)}_{\mathcal{O}\left(\kappa_s^2\, a_f^0(\beta)\right)} - \underbrace{S_{f,eff,1}(n_t,\kappa,\beta,\mu)}_{\mathcal{O}\left(\kappa_s^2\, a_f^1(\beta)\right) - \mathcal{O}\left(\kappa_s^2\, a_f^{n_t-1}(\beta)\right)} - \underbrace{S_{f,eff,2}(n_t,\kappa,\beta,\mu)}_{\mathcal{O}\left(\kappa_s^2\, a_f^{n_t}(\beta)\right)}$$

depends only on $L_i = \mathrm{tr}_c(P_{\bar{x}_i})$ and $L_i^* = \mathrm{tr}_c(P_{\bar{x}_i}^{\dagger})$.

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- Mean field action for effective nearest-neighbor Polyakov loop model:

$$S_{mf}\left(L_a, L_a^*, \bar{L}, \bar{L}^*\right) = \frac{1}{V}\sum_i^V \left\{ \left(L_a - \bar{L}\right) \frac{\partial S_{eff,tot}(L_1,\ldots,L_V,L_1^*,\ldots,L_V^*)}{\partial L_i}\bigg|_{\substack{L_i=\bar{L}\\ L_i^*=\bar{L}^*}} + \left(L_a^* - \bar{L}^*\right) \frac{\partial S_{eff,tot}(L_1,\ldots,L_V,L_1^*,\ldots,L_V^*)}{\partial L_i^*}\bigg|_{\substack{L_i=\bar{L}\\ L_i^*=\bar{L}^*}} \right\},$$

Complex fermion determinant $\longrightarrow \bar{L} = \langle L \rangle \neq \langle L^* \rangle = \bar{L}^*$.
To avoid subtleties (c.f. [Fukushima & Hidaka 2007, arXiv:hep-ph/0610323]) :

- define mean field in phase quenched system:

$$\bar{L} = \langle L \rangle_q \left(\bar{L}\right) = \frac{1}{Z_{s,q}\left(\bar{L}\right)} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \left\{ L(\theta_1,\theta_2)\, H(\theta_1,\theta_2) \cdot \left| \mathrm{Det}(D(\theta_1,\theta_2))\, \mathrm{e}^{-S_{mf}\left(L(\theta_1,\theta_2),\bar{L}\right)} \right| \right\},$$

where

$$Z_{s,q}\left(\bar{L}\right) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \left\{ H(\theta_1,\theta_2) \left| \mathrm{Det}(D(\theta_1,\theta_2))\, \mathrm{e}^{-S_{mf}\left(L(\theta_1,\theta_2),\bar{L}\right)} \right| \right\},$$

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To avoid subtleties (c.f. [Fukushima & Hidaka 2007, arXiv:hep-ph/0610323]) :

- define mean field in phase quenched system:
- use reweighting (exact for spatially localized observables):

$$\langle L\rangle\left(\bar{L}\right) = \frac{1}{Z_s\left(\bar{L}\right)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta_1\mathrm{d}\theta_2\left\{L(\theta_1,\theta_2)\,H(\theta_1,\theta_2)\,\mathrm{Det}\left(D(\theta_1,\theta_2)\right)\,\mathrm{e}^{-S_{mf}\left(L(\theta_1,\theta_2),\bar{L}\right)}\right\},$$

where

$$Z_s\left(\bar{L}\right) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta_1\mathrm{d}\theta_2\left\{H(\theta_1,\theta_2)\,\mathrm{Det}\left(D(\theta_1,\theta_2)\right)\,\mathrm{e}^{-S_{mf}\left(L(\theta_1,\theta_2),\bar{L}\right)}\right\},$$

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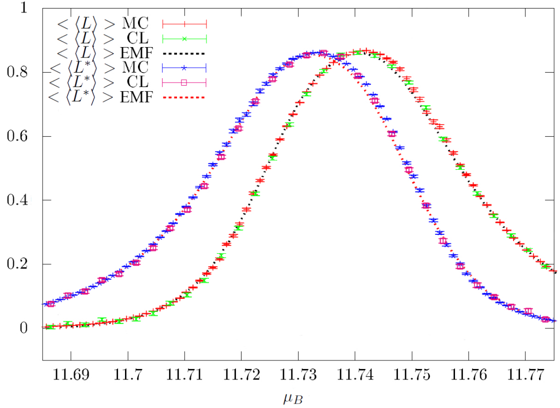
- Comparison of mean-field result with Monte Carlo and Complex Langevin [Langelage et. al. 2014, arXiv:1403.4162]:

($\kappa = 0.01$, $N_t = 200$ and $\beta = 5.7$)



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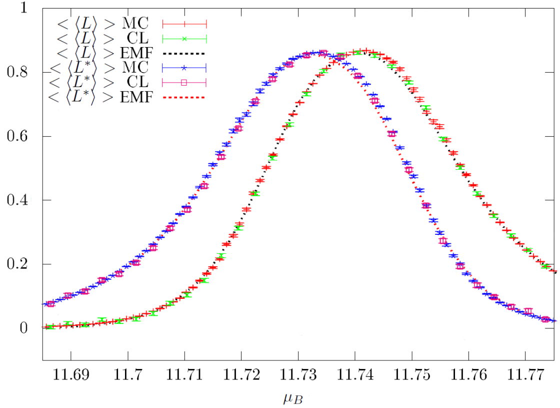

- Mean-field results agree very well with Monte Carlo estimates!

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2. Isospin QCD at finite density

- Mass-degenerate two-flavor isospin QCD:

+ no sign problem: $\mathrm{Det}(M(\mu))\,\mathrm{Det}(M(-\mu)) = \mathrm{Det}(M(\mu))\,\mathrm{Det}(M(\mu))^* = |\mathrm{Det}(M(\mu))|^2 \in \mathbb{R}^+$.

— but: other difficulties after pion condensation at $\mu > m_\pi/2$:

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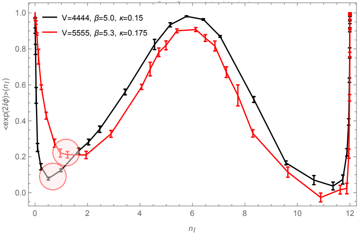
− but: other difficulties after pion condensation at $\mu > m_{\pi}/2$:

  - Lattice saturation effects become relevant quickly.

average sign $\langle R \rangle_{|QCD|}$ vs. n_I



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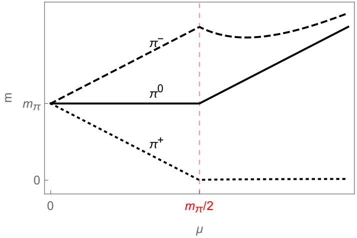
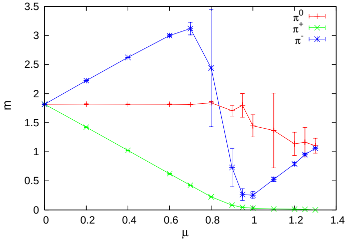
- Measuring mass spectra vs. μ difficult.

$\mathrm{SU}(2)$ principal chiral model
[Son & Stephanov, 2001]



lattice isospin QCD



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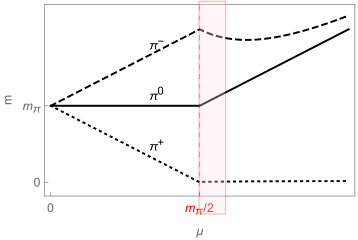
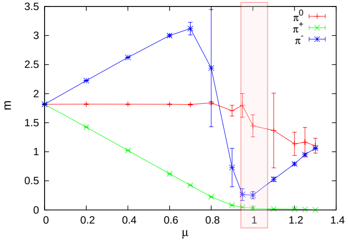


lattice isospin QCD



→ Why no agreement for $m_\pi/2 < \mu < m_\rho/2$?

- Effective Lagrangian cf. [Son & Stephanov, 2001]

$$\mathcal{L}_{eff} = -\frac{f_\pi^2}{4}\,\mathrm{tr}\big[\big(\partial_\rho\Sigma - \mathrm{i}\,\mu\,\delta_{\rho,0}\big(\tau_3\Sigma - \Sigma\tau_3\big)\big)\,g^{\rho\nu}\,\big(\partial_\nu\Sigma^\dagger - \mathrm{i}\,\mu\,\delta_{\nu,0}\big(\tau_3\Sigma^\dagger - \Sigma^\dagger\tau_3\big)\big)\big] - \frac{1}{4}\,\mathrm{tr}\big[S^\dagger\Sigma + \Sigma^\dagger S\big]$$

with:

$$\Sigma = \Sigma(\overline{\pi}) = \mathbb{1}\sqrt{1 - \frac{\|\overline{\pi}\|^2}{f_\pi^2}} + \mathrm{i}\,\frac{\overline{\tau}\cdot\overline{\pi}}{f_\pi}\,,$$

$$S = \mathbb{1}\,s^4 + \mathrm{i}\,\overline{\tau}\cdot\overline{s} \qquad (s^4 = f_\pi^2 m_\pi^2 \sim 2\,m_q).$$

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with:

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- Saddle point approximation:

→ write $\overline{\pi} = \overline{\pi}_0 + \eta(\overline{\pi}_0, \overline{\pi}(x))$,

$$\eta^i(\overline{\pi}_0,\overline{\pi}) = \pi^i - \tfrac{1}{2}\Gamma^i_{kl}(\overline{\pi}_0)\,\pi^k\,\pi^l + \mathcal{O}\big(\big(\pi^l(x)\big)^3\big)$$

$$\Gamma^i_{kl}(\overline{\pi}) = \tfrac{1}{2}\,h^{ij}(\overline{\pi})\left(\frac{\partial h_{jk}(\overline{\pi})}{\partial\pi^l} + \frac{\partial h_{jl}(\overline{\pi})}{\partial\pi^k} - \frac{\partial h_{kl}(\overline{\pi})}{\partial\pi^j}\right)$$

$$h_{ij}(\overline{\pi}) = \tfrac{1}{2}\,\mathrm{tr}\left[\frac{\partial\Sigma^\dagger(\overline{\pi})}{\partial\pi^i}\,\frac{\partial\Sigma(\overline{\pi})}{\partial\pi^j}\right]$$

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$$\mathcal{L}_{eff} = -\frac{f_\pi^2}{4}\,\mathrm{tr}\big[\big(\partial_\rho\Sigma - \mathrm{i}\,\mu\,\delta_{\rho,0}\big(\tau_3\Sigma - \Sigma\tau_3\big)\big)\,g^{\rho\nu}\,\big(\partial_\nu\Sigma^\dagger - \mathrm{i}\,\mu\,\delta_{\nu,0}\big(\tau_3\Sigma^\dagger - \Sigma^\dagger\tau_3\big)\big)\big] - \frac{1}{4}\,\mathrm{tr}\big[S^\dagger\Sigma + \Sigma^\dagger S\big]$$

with:

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$$\eta^i(\bar{\pi}_0,\bar{\pi}) = \pi^i - \tfrac{1}{2}\Gamma^i_{kl}(\bar{\pi}_0)\,\pi^k\,\pi^l + \mathcal{O}\big((\pi^l(x))^3\big)$$

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- Saddle point approximation:

→ write $\bar{\pi} = \bar{\pi}_0 + \bar{\eta}(\bar{\pi}_0,\bar{\pi}(x))$,

→ $\bar{\pi}(x) = 0 \rightarrow$ effective potential: $\boxed{V_{eff}(\bar{\pi}_0)}$, min. $\rightarrow$ vacuum $\bar{\pi}_0$,

→ Euler-Lagrange eqn. for $\bar{\pi}(x)$ + Fourier trans. $\rightarrow$ e.o.m.: $\boxed{M^i{}_j(\bar{\pi}_0,p)\,\hat{\pi}^j(\bar{\pi}_0,p) = 0}$,

- $\det M(\bar{\pi}_0,p) = 0 \rightarrow$ 3 dispersion relations: $p^0 = p^0_{(i)}(\bar{p})$, $i = 1,2,3$.
- zero-eigenvector of $M(\bar{\pi}_0,(p^0_{(i)}(\bar{p}),\bar{p})) \rightarrow$ state vector corresp. to i^{th} dispersion relation.

$\mu < m_\pi/2$

vacuum: $\bar{\pi}_0 = 0$

$$M(0,p) = \begin{pmatrix} E^2-\|\bar{p}\|^2-m_\pi^2+4\mu^2 & 4\,\mathrm{i}\,E\,\mu & 0 \\ -4\,\mathrm{i}\,E\,\mu & E^2-\|\bar{p}\|^2-m_\pi^2+4\mu^2 & 0 \\ 0 & 0 & E^2-\|\bar{p}\|^2-m_\pi^2 \end{pmatrix}$$

state vectors and dispersion relations:

$$\begin{pmatrix} \widehat{\pi}^1 \\ \widehat{\pi}^2 \\ \widehat{\pi}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{m_\pi^2 + \|\bar{p}\|^2}$$

$$\begin{pmatrix} \widehat{\pi}^1 \\ \widehat{\pi}^2 \\ \widehat{\pi}^3 \end{pmatrix} = \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{m_\pi^2 + \|\bar{p}\|^2} + 2\,\mu$$

$$\begin{pmatrix} \widehat{\pi}^1 \\ \widehat{\pi}^2 \\ \widehat{\pi}^3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{m_\pi^2 + \|\bar{p}\|^2} - 2\,\mu$$

$\mu > m_\pi/2$

vacuum: $\bar{\pi}_0 = \begin{pmatrix} \pi_0^r \cos(\phi_0) \\ \pi_0^r \sin(\phi_0) \\ 0 \end{pmatrix}$,

with $\pi_0^r = \begin{cases} 0 & \text{if} \quad \mu \leq m_\pi/2 \\ f_\pi \sqrt{1 - \frac{m_\pi^4}{16\mu^4}} & \text{if} \quad \mu > m_\pi/2 \end{cases}$

$$M'(\bar{\pi}_0,p) = \begin{pmatrix} E^2-\|\bar{p}\|^2-4\mu^2+\frac{m_\pi^4}{4\mu^2} & -\frac{\mathrm{i}\,E\,m_\pi^2}{\mu} & 0 \\ \frac{\mathrm{i}\,E\,m_\pi^2}{\mu} & E^2-\|\bar{p}\|^2 & 0 \\ 0 & 0 & E^2-\|\bar{p}\|^2-4\mu^2 \end{pmatrix}$$

state vectors and dispersion relations:

$$\begin{pmatrix} \widehat{\pi'}^1 \\ \widehat{\pi'}^2 \\ \widehat{\pi'}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{\|\bar{p}\|^2 + 4\,\mu^2}$$

$$\begin{pmatrix} \widehat{\pi'}^1 \\ \widehat{\pi'}^2 \\ \widehat{\pi'}^3 \end{pmatrix} = \begin{pmatrix} -\frac{2\,\mathrm{i}\,\mu\,E}{\sqrt{7\,m_\pi^4+16\,\mu^4}} \\ \frac{2\left(\mu^2\,C_-(\bar{p})-m_\pi^4\right)}{m_\pi^2\,\sqrt{7\,m_\pi^4+16\,\mu^4}} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{\|\bar{p}\|^2 + C_+(\bar{p})}$$

$$\begin{pmatrix} \widehat{\pi'}^1 \\ \widehat{\pi'}^2 \\ \widehat{\pi'}^3 \end{pmatrix} = \begin{pmatrix} -\frac{4\,\mathrm{i}\,\mu\,E}{16\mu^4-m_\pi^4} \\ \frac{4\left(\mu^2\,C_+(\bar{p})-m_\pi^4\right)}{16\mu^4-m_\pi^4} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{\|\bar{p}\|^2 + C_-(\bar{p})}$$

$$C_\pm(\bar{p}) = \frac{D \pm \sqrt{D^2+64\,m_\pi^4\,\mu^2\,\|\bar{p}\|^2}}{8\,\mu^2} ,$$

$$D = 3\,m_\pi^4 + 16\,\mu^4$$

$\mu < m_\pi/2$	$\mu > m_\pi/2$
vacuum: $\bar{\pi}_0 = 0$	vacuum: $\bar{\pi}_0 = \begin{pmatrix} \pi_0^r \cos(\phi_0) \\ \pi_0^r \sin(\phi_0) \\ 0 \end{pmatrix}$, with $\pi_0^r = \begin{cases} 0 & \text{if } \mu \leq m_\pi/2 \\ f_\pi \sqrt{1 - \frac{m_\pi^4}{16\mu^4}} & \text{if } \mu > m_\pi/2 \end{cases}$
$M(0,p) = \begin{pmatrix} E^2-\lVert\bar{p}\rVert^2-m_\pi^2+4\mu^2 & 4\,\mathrm{i}\,E\,\mu & 0 \\ -4\,\mathrm{i}\,E\,\mu & E^2-\lVert\bar{p}\rVert^2-m_\pi^2+4\mu^2 & 0 \\ 0 & 0 & E^2-\lVert\bar{p}\rVert^2-m_\pi^2 \end{pmatrix}$	$M'(\bar{\pi}_0,p) = \begin{pmatrix} E^2-\lVert\bar{p}\rVert^2-4\mu^2+\frac{m_\pi^4}{4\mu^2} & -\frac{\mathrm{i}\,E\,m_\pi^2}{\mu} & 0 \\ \frac{\mathrm{i}\,E\,m_\pi^2}{\mu} & E^2-\lVert\bar{p}\rVert^2 & 0 \\ 0 & 0 & E^2-\lVert\bar{p}\rVert^2-4\mu^2 \end{pmatrix}$

state vectors and dispersion relations ($\mu < m_\pi/2$):

$$\begin{pmatrix} \widehat{\pi}^1 \\ \widehat{\pi}^2 \\ \widehat{\pi}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{m_\pi^2 + \lVert\bar{p}\rVert^2}$$

$$\begin{pmatrix} \widehat{\pi}^1 \\ \widehat{\pi}^2 \\ \widehat{\pi}^3 \end{pmatrix} = \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{m_\pi^2 + \lVert\bar{p}\rVert^2} + 2\mu$$

$$\begin{pmatrix} \widehat{\pi}^1 \\ \widehat{\pi}^2 \\ \widehat{\pi}^3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{m_\pi^2 + \lVert\bar{p}\rVert^2} - 2\mu$$

state vectors and dispersion relations ($\mu > m_\pi/2$):

$$\begin{pmatrix} \widehat{\pi'}^1 \\ \widehat{\pi'}^2 \\ \widehat{\pi'}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{\lVert\bar{p}\rVert^2 + 4\mu^2}$$

$$\begin{pmatrix} \widehat{\pi'}^1 \\ \widehat{\pi'}^2 \\ \widehat{\pi'}^3 \end{pmatrix} = \begin{pmatrix} -\frac{2\,\mathrm{i}\,\mu\,E}{\sqrt{7m_\pi^4+16\mu^4}} \\ \frac{2\left(\mu^2 C_-(\bar{p})-m_\pi^4\right)}{m_\pi^2\sqrt{7m_\pi^4+16\mu^4}} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{\lVert\bar{p}\rVert^2 + C_+(\bar{p})}$$

$$\begin{pmatrix} \widehat{\pi'}^1 \\ \widehat{\pi'}^2 \\ \widehat{\pi'}^3 \end{pmatrix} = \begin{pmatrix} -\frac{4\,\mathrm{i}\,\mu\,E}{16\mu^4-m_\pi^4} \\ \frac{4\left(\mu^2 C_+(\bar{p})-m_\pi^4\right)}{16\mu^4-m_\pi^4} \\ 0 \end{pmatrix} \quad \text{iff} \quad E = \pm\sqrt{\lVert\bar{p}\rVert^2 + C_-(\bar{p})}$$

momentum dependent!

$$C_\pm(\bar{p}) = \frac{D \pm \sqrt{D^2 + 64\,m_\pi^4\,\mu^2\,\lVert\bar{p}\rVert^2}}{8\mu^2},$$

$$D = 3m_\pi^4 + 16\mu^4$$

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$\mu < m_\pi/2$	$\mu > m_\pi/2$

$\mu < m_\pi/2$

state vectors and effective masses:

$$\begin{pmatrix}\hat{\pi}^1\\\hat{\pi}^2\\\hat{\pi}^3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad \text{iff}\quad m=\pm m_\pi$$

$$\begin{pmatrix}\hat{\pi}^1\\\hat{\pi}^2\\\hat{\pi}^3\end{pmatrix}=\begin{pmatrix}1\\\mathrm{i}\\0\end{pmatrix}\quad \text{iff}\quad m=\pm m_\pi+2\mu$$

$$\begin{pmatrix}\hat{\pi}^1\\\hat{\pi}^2\\\hat{\pi}^3\end{pmatrix}=\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix}\quad \text{iff}\quad m=\pm m_\pi-2\mu$$

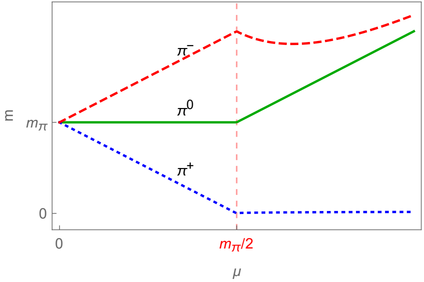
$\mu > m_\pi/2$

state vectors and effective masses:

$$\begin{pmatrix}\hat{\pi}_\alpha\\\hat{\pi}_\phi\\\hat{\pi}_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad \text{iff}\quad m=\pm 2\mu$$

$$\begin{pmatrix}\hat{\pi}_\alpha\\\hat{\pi}_\phi\\\hat{\pi}_3\end{pmatrix}=\begin{pmatrix}\pm\mathrm{i}\sqrt{1-\frac{4m_\pi^4}{7m_\pi^4+16\mu^4}}\\\frac{2}{\sqrt{7+\frac{16\mu^4}{m_\pi^4}}}\\0\end{pmatrix}\quad \text{iff}\quad m=\pm\frac{\sqrt{3m_\pi^4+16\mu^4}}{2\mu}$$

$$\begin{pmatrix}\hat{\pi}_\alpha\\\hat{\pi}_\phi\\\hat{\pi}_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad \text{iff}\quad m=0$$



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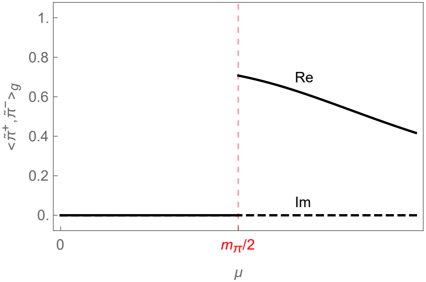
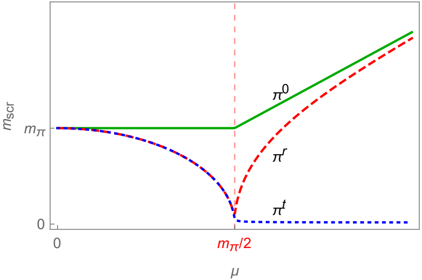
$\mu < m_\pi/2$	$\mu > m_\pi/2$
state vectors and effective masses:	state vectors and effective masses:
$\begin{pmatrix}\hat{\pi}^1\\ \hat{\pi}^2\\ \hat{\pi}^3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ iff $m = \pm m_\pi$	$\begin{pmatrix}\hat{\pi}_\alpha\\ \hat{\pi}_\phi\\ \hat{\pi}_3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ iff $m = \pm 2\mu$
$\begin{pmatrix}\hat{\pi}^1\\ \hat{\pi}^2\\ \hat{\pi}^3\end{pmatrix} = \begin{pmatrix}1\\ \mathrm{i}\\0\end{pmatrix}$ iff $m = \pm m_\pi + 2\mu$	$\begin{pmatrix}\hat{\pi}_\alpha\\ \hat{\pi}_\phi\\ \hat{\pi}_3\end{pmatrix} = \begin{pmatrix}\pm\mathrm{i}\sqrt{1-\frac{4m_\pi^4}{7m_\pi^4+16\mu^4}}\\ \frac{2}{\sqrt{7+\frac{16\mu^4}{m_\pi^4}}}\\ 0\end{pmatrix}$ iff $m = \pm\frac{\sqrt{3m_\pi^4+16\mu^4}}{2\mu}$
$\begin{pmatrix}\hat{\pi}^1\\ \hat{\pi}^2\\ \hat{\pi}^3\end{pmatrix} = \begin{pmatrix}1\\ -\mathrm{i}\\0\end{pmatrix}$ iff $m = \pm m_\pi - 2\mu$	$\begin{pmatrix}\hat{\pi}_\alpha\\ \hat{\pi}_\phi\\ \hat{\pi}_3\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ iff $m = 0$
	non-orthogonal!



2. Isospin QCD at finite density

$\mu < m_\pi/2$	$\mu > m_\pi/2$
state vectors and **screening masses**:	state vectors and **screening masses**:
$\begin{pmatrix}\hat\pi^1\\ \hat\pi^2\\ \hat\pi^3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ iff $m_{scr} = \pm m_\pi$	$\begin{pmatrix}\hat\pi_\alpha\\ \hat\pi_\phi\\ \hat\pi_3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ iff $m_{scr} = \pm 2\mu$
$\begin{pmatrix}\hat\pi^1\\ \hat\pi^2\\ \hat\pi^3\end{pmatrix} = \begin{pmatrix}1\\ \mathrm{i}\\0\end{pmatrix}$ iff $m_{scr} = \pm\sqrt{m_\pi^2 - (2\mu)^2}$	$\begin{pmatrix}\hat\pi_\alpha\\ \hat\pi_\phi\\ \hat\pi_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ iff $m_{scr} = \pm\sqrt{(2\mu)^2 - \frac{m_\pi^4}{(2\mu)^2}}$
$\begin{pmatrix}\hat\pi^1\\ \hat\pi^2\\ \hat\pi^3\end{pmatrix} = \begin{pmatrix}1\\ -\mathrm{i}\\0\end{pmatrix}$ iff $m_{scr} = \pm\sqrt{m_\pi^2 - (2\mu)^2}$	$\begin{pmatrix}\hat\pi_\alpha\\ \hat\pi_\phi\\ \hat\pi_3\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ iff $m_{scr} = 0$



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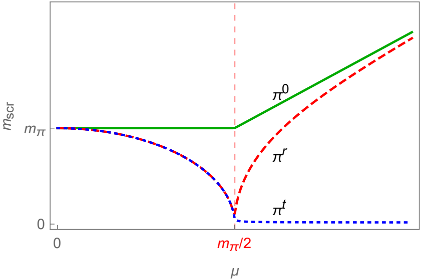
$\mu < m_\pi/2$	$\mu > m_\pi/2$
state vectors and **screening masses**:	state vectors and **screening masses**:
$\begin{pmatrix}\hat\pi^1\\\hat\pi^2\\\hat\pi^3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ iff $m_{scr}=\pm m_\pi$	$\begin{pmatrix}\hat\pi_\alpha\\\hat\pi_\phi\\\hat\pi_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ iff $m_{scr}=\pm 2\mu$
$\begin{pmatrix}\hat\pi^1\\\hat\pi^2\\\hat\pi^3\end{pmatrix}=\begin{pmatrix}1\\i\\0\end{pmatrix}$ iff $m_{scr}=\pm\sqrt{m_\pi^2-(2\mu)^2}$	$\begin{pmatrix}\hat\pi_\alpha\\\hat\pi_\phi\\\hat\pi_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ iff $m_{scr}=\pm\sqrt{(2\mu)^2-\frac{m_\pi^4}{(2\mu)^2}}$
$\begin{pmatrix}\hat\pi^1\\\hat\pi^2\\\hat\pi^3\end{pmatrix}=\begin{pmatrix}1\\-i\\0\end{pmatrix}$ iff $m_{scr}=\pm\sqrt{m_\pi^2-(2\mu)^2}$	$\begin{pmatrix}\hat\pi_\alpha\\\hat\pi_\phi\\\hat\pi_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ iff $m_{scr}=0$
	orthogonal!



- Lattice action:

$$S = \frac{1}{4}\sum_{x}\Big\{-\kappa\sum_{\nu=1}^{d}\mathrm{tr}\big[\Sigma_x^\dagger \mathrm{e}^{\mu\,\sigma_3\,\delta_{\nu,d}}\,\Sigma_{x+\hat{\nu}}\,\mathrm{e}^{-\mu\,\sigma_3\,\delta_{\nu,d}} + \Sigma_x^\dagger\,\mathrm{e}^{-\mu\,\sigma_3\,\delta_{\nu,d}}\,\Sigma_{x-\hat{\nu}}\,\mathrm{e}^{\mu\,\sigma_3\,\delta_{\nu,d}}\big] - \mathrm{tr}\big[\Sigma_x^\dagger\,S + S^\dagger\,\Sigma_x\big]\Big\} + \text{const.}$$

where

$f_\pi^2 a^{d-2} \to \kappa$

$a\,\mu \to \mu$

$a^d\,S \to S = \mathbb{1}\cdot s^4 + \mathrm{i}\,\bar{\tau}\cdot\bar{s}$

$\pi_x^i / f_\pi \to \pi_x^i\,,\quad$ s.t. $\quad \Sigma_x = \Sigma(\bar{\pi}_x) = \pi_x^4\,\mathbb{1} + \mathrm{i}\,\bar{\pi}_x\cdot\bar{\tau}$, with $(\pi_x^4)^2 + \|\bar{\pi}_x\|^2 = 1$.

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- Lattice action:

$$S = \frac{1}{4}\sum_x \Big\{ -\kappa \sum_{\nu=1}^{d} \mathrm{tr}\big[\Sigma_x^\dagger e^{\mu\sigma_3\delta_{\nu,d}}\Sigma_{x+\hat{\nu}} e^{-\mu\sigma_3\delta_{\nu,d}} + \Sigma_x^\dagger e^{-\mu\sigma_3\delta_{\nu,d}}\Sigma_{x-\hat{\nu}} e^{\mu\sigma_3\delta_{\nu,d}}\big] - \mathrm{tr}\big[\Sigma_x^\dagger S + S^\dagger \Sigma_x\big]\Big\} + \text{const.}$$

where

$f_\pi^2 a^{d-2} \to \kappa$

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$a^d\, S \to S = \mathbb{1}\cdot s^4 + \mathrm{i}\,\bar{\tau}\cdot\bar{s}$

$\pi_x^i / f_\pi \to \pi_x^i$, s.t. $\Sigma_x = \Sigma(\bar{\pi}_x) = \pi_x^4\,\mathbb{1} + \mathrm{i}\,\bar{\pi}_x\cdot\bar{\tau}$, with $(\pi_x^4)^2 + \|\bar{\pi}_x\|^2 = 1$.

- action in general complex for non-zero values of μ.

$\to$ partition function $Z = \int \mathcal{D}[\Sigma]\, e^{-S[\Sigma]}$ has sign problem!

$\to$ can be overcome by changing to dual, so-called "flux variable" representation.

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3. Lattice $\mathrm{SU}(2)$ principal chiral model at finite density

- Lattice action:

$$S = \frac{1}{4}\sum_x \Big\{ -\kappa \sum_{\nu=1}^{d} \mathrm{tr}\big[\Sigma_x^{\dagger}\, e^{\mu\sigma_3\delta_{\nu,d}}\, \Sigma_{x+\hat{\nu}}\, e^{-\mu\sigma_3\delta_{\nu,d}} + \Sigma_x^{\dagger}\, e^{-\mu\sigma_3\delta_{\nu,d}}\, \Sigma_{x-\hat{\nu}}\, e^{\mu\sigma_3\delta_{\nu,d}} \big] - \mathrm{tr}\big[\Sigma_x^{\dagger} S + S^{\dagger} \Sigma_x \big] \Big\} + \text{const.}$$

where

$$f_\pi^2 a^{d-2} \to \kappa$$
$$a\,\mu \to \mu$$
$$a^d\, S \to S = \mathbb{1}\cdot s^4 + \mathrm{i}\,\bar{\tau}\cdot\bar{s}$$
$$\pi_x^i / f_\pi \to \pi_x^i\,, \quad \text{s.t.} \quad \Sigma_x = \Sigma(\bar{\pi}_x) = \pi_x^4\,\mathbb{1} + \mathrm{i}\,\bar{\pi}_x\cdot\bar{\tau}\,, \text{ with } (\pi_x^4)^2 + \|\bar{\pi}_x\|^2 = 1\,.$$

- action in general complex for non-zero values of μ.

  $\rightarrow$ partition function $Z = \int \mathcal{D}[\Sigma]\, e^{-S[\Sigma]}$ has sign problem!

  $\rightarrow$ can be overcome by changing to dual, so-called "flux variable" representation.

- Toy model for testing methods to extract mass spectrum in isospin lattice QCD (symmetry integrated out analytically, need for source terms, ...).

- Basics of "flux-variable" dualization:

$\rightarrow$ Boltzmann factor: $\mathrm{e}^{\kappa\sum_{x,\nu}\phi_x\phi_{x+\hat{\nu}}+s\sum_x\phi_x}$,

$\rightarrow$ factorize: $\left(\prod_{x,\nu}\mathrm{e}^{\kappa\phi_x\phi_{x+\hat{\nu}}}\right)\left(\prod_x\mathrm{e}^{s\phi_x}\right)$,

$\rightarrow$ Taylorexpand: $\mathrm{e}^{\kappa\phi_x\phi_{x+\hat{\nu}}}=\sum_{\xi_{x,\nu}=0}^{\infty}\frac{\left(\kappa\phi_x\phi_{x+\hat{\nu}}\right)^{\xi_{x,\nu}}}{\xi_{x,\nu}!}$, $\mathrm{e}^{s\phi_x}=\sum_{n_x=0}^{\infty}\frac{(s\phi_x)^{n_x}}{n_x!}$

$\rightarrow$ expand product of sums:

$$\sum_{\{\xi,n\}}\prod_x\left(\prod_\nu\frac{\kappa^{\xi_{x,\nu}}}{\xi_{x,\nu}!}\right)\frac{s^{n_x}}{n_x!}\,\phi_x^{n_x+\sum_\nu\left(\xi_{x,\nu}+\xi_{x-\hat{\nu},\nu}\right)}$$

$\rightarrow$ integrate out the original variables ϕ:

$$Z=\sum_{\{\xi,n\}}\prod_x\left(\prod_\nu\frac{\kappa^{\xi_{x,\nu}}}{\xi_{x,\nu}!}\right)\frac{s^{n_x}}{n_x!}\,W\Big(n_x+\sum_\nu(\xi_{x,\nu}+\xi_{x-\hat{\nu},\nu})\Big)$$

$\rightarrow$ flux variable: $\xi_{x,\nu}$,

$\rightarrow$ monomer number: n_x .

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3. Lattice $\mathrm{SU}(2)$ principal chiral model at finite density

- Dual rep. of $\mathrm{SU}(2)$ principal chiral model: cf. [Bruckmann et al., 2015],[Forcand & Rindlisbacher, 2015]

$$Z = \sum_{\{k,l,\xi^{(i)},\chi,p,q,n^{(3)},\ldots,n^{(N)}\}} \Bigg\{ \prod_{x,\nu} \frac{\kappa^{|k_{x,\nu}|+2l_{x,\nu}+\sum_{i=3}^{N}\chi_{x,\nu}^{(i)}}}{(|k_{x,\nu}|+l_{x,\nu})!\,l_{x,\nu}!\,\prod_{i=3}^{N}\chi_{x,\nu}^{(i)}!} \Bigg\}$$

$$\Bigg\{ \prod_{x} \frac{\mathrm{e}^{2\mu k_{x,d}}\,\mathrm{e}^{\mathrm{i}\phi_{s,x}p_x}\,s^{|p_x|+2q_x}\prod_{i=3}^{N}(s^i)^{n_x^3}}{2^{(|p_x|+2q_x)/2}(|p_x|+q_x)!\,q_x!\,\prod_{i=3}^{N}n_x^{(i)}!}\,\delta\Big(p_x+\sum_{\nu}(k_{x,\nu}-k_{x-\hat{\nu},\nu})\Big)$$

$$W\Big(A_x+|p_x|+2q_x, C_x^{(3)}+n_x^{(3)},\ldots,C_x^{(N)}+n_x^{(N)}\Big)\Bigg\}$$

with $A_x = \sum_{\nu}(|k_{x,\nu}|+|k_{x-\hat{\nu},\nu}|+2(l_{x,\nu}+l_{x-\hat{\nu},\nu}))$, $C_x^{(i)} = \sum_{\nu}\Big(\chi_{x,\nu}^{(i)}+\chi_{x-\hat{\nu},\nu}^{(i)}\Big)$ and

$$W(A,C^{(3)},\ldots,C^{(N)}) = \frac{\Gamma\big(\frac{2+A}{2}\big)\prod_{i=3}^{N}\frac{1+(-1)^{C^{(i)}}}{2}\Gamma\big(\frac{1+C^{(i)}}{2}\big)}{2^{(2+A)/2}\Gamma\big(\frac{N+A+\sum_{i=3}^{N}C^{(i)}}{2}\big)}$$

→ **No sign problem!**

- Gauss constraint for k-variables and Evenness constraints for χ-variables!

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- Variables $l_{x,\nu}$ and q_x not subject to constraint: can be sampled by ordinary local Metropolis .

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- Variables $l_{x,\nu}$ and q_x not subject to constraint: can be sampled by ordinary local Metropolis .
- Due to constraints: cannot use ordinary local Metropolis to sample $k_{x,\nu}$, p_x and $\chi^{(i)}_{x,\nu}$, $n^{(i)}_x$ variables.

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- Variables $l_{x,\nu}$ and q_x not subject to constraint: can be sampled by ordinary local Metropolis .
- Due to constraints: cannot use ordinary local Metropolis to sample $k_{x,\nu}$, p_x and $\chi_{x,\nu}^{(i)}$, $n_x^{(i)}$ variables.

$\rightarrow$ Instead of looking at Z only,

$$Z = \sum_{\{k,l,\chi^{(i)},p,q,n^{(i)}\}} \left\{ \prod_{x,\nu} \frac{\kappa^{|k_{x,\nu}|+2l_{x,\nu}+\sum_{i=3}^{4}\chi_{x,\nu}^{(i)}}}{(|k_{x,\nu}|+l_{x,\nu})!\,l_{x,\nu}!\,\prod\limits_{i=3}^{4}\chi_{x,\nu}^{(i)}!} \right\}$$

$$\left\{ \prod_{x} \frac{\mathrm{e}^{2\mu k_{x,d}}\,\mathrm{e}^{\mathrm{i}\phi_{s,x}p_x}\,s^{|p_x|+2q_x}\prod\limits_{i=3}^{4}(s^i)^{n_x^3}}{2^{(|p_x|+2q_x)/2}(|p_x|+q_x)!\,q_x!\,\prod\limits_{i=3}^{4}n_x^{(i)}!}\,\delta\Big(p_x+\sum_{\nu}(k_{x,\nu}-k_{x-\widehat{\nu},\nu})\Big)\right.$$

$$\left. W\Big(A_x+|p_x|+2q_x, C_x^{(3)}+n_x^{(3)}, C_x^{(4)}+n_x^{(N)}\Big)\right\}$$

- Variables $l_{x,\nu}$ and q_x not subject to constraint: can be sampled by ordinary local Metropolis .
- Due to constraints: cannot use ordinary local Metropolis to sample $k_{x,\nu}$, p_x and $\chi_{x,\nu}^{(i)}$, $n_x^{(i)}$ variables.

→ Instead of looking at Z only,

→ look also at e.g. $Z_2^{21}(x,y) = \frac{\partial^2 Z}{\partial s_x^+ \partial s_y^-} = Z \cdot \langle \pi^+(x) \pi^-(y) \rangle$

$$\frac{\partial^2 Z}{\partial s_x^{\pm} \partial s_y^{\mp}} = \sum_{\{k,l,\chi^{(i)},p,q,n^{(i)}\}} \left\{ \prod_{z,\nu} \frac{\kappa^{|k_{z,\nu}|+2 l_{z,\nu}+\sum_{i=3}^{4} \chi_{z,\nu}^{(i)}}}{(|k_{z,\nu}|+l_{z,\nu})!\, l_{z,\nu}!\, \prod_{i=3}^{4} \chi_{z,\nu}^{(i)}!} \right\}$$

$$\left\{ \prod_{z} \frac{s_z^{|p_z|+2q_z}\, e^{i\phi_{s,x} p_z}\, e^{2\mu k_{z,d}} \prod_{i=3}^{4} (s_z^i)^{n_z^{(i)}}}{2^{(|p_z|+2q_z)/2} (|p_z|+q_z)!\, q_z!\, \prod_{i=3}^{4} n_z^{(i)}!} \delta\Big(p_z \pm \delta_{x,z} \mp \delta_{y,z} + \sum_{\nu} (k_{z,\nu} - k_{z-\hat{\nu},\nu})\Big) \right.$$

$$\left. W\Big(A_z + |p_z| + 2q_z + \delta_{x,z} + \delta_{x,z}, C_z^{(3)} + n_z^{(3)}, C_z^{(4)} + n_z^{(4)}\Big) \right\}$$

- Define generalized partition function Z_{gen}:

$$Z_{gen} = Z + \sum_{i=1}^{4} c_i \sum_{x} Z_1^i(x) + \sum_{i,j=1}^{4} c_{ij} \sum_{x,y} Z_2^{ij}(x,y) \, (+ \ldots)$$

with the *a priori weights* $\{ c_i \}_{i=1,2}$ and $\{ c_{ij} \}_{i,j=1,2}$ and :

- $Z_2^{ij}(x,y) = \frac{\partial^2 Z}{\partial \tilde{s}_x^i \partial \tilde{s}_y^j} = Z \cdot \langle \tilde{\pi}^i(x) \tilde{\pi}^j(y) \rangle$,
- $Z_1^i(x) = \frac{\partial Z}{\partial \tilde{s}_x^i} = Z \cdot \langle \tilde{\pi}^i(x) \rangle$,

with $\tilde{s}_x = \left(s_x^-, s_x^+, s^3, s^4 \right)$ and $\tilde{\pi}_x = \left(\pi_x^-, \pi_x^+, \pi_x^3, \pi_x^4 \right)$.

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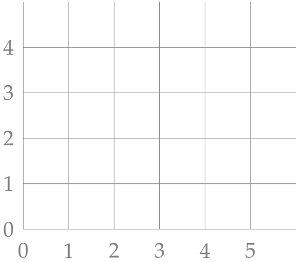
- Define generalized partition function Z_{gen}:

$$Z_{gen} = Z + \sum_{i=1}^{4} c_i \sum_{x} Z_1^i(x) + \sum_{i,j=1}^{4} c_{ij} \sum_{x,y} Z_2^{ij}(x,y) \, (+\ldots)$$

- Sample configurations for Z_{gen} with *generalized worm algorithm* . [Prokof'ev & Svistunov, 2001]

config. C contributing to Z_{gen} :



config. weight $W[C]$:

$$W[C] \propto \prod_{z} \Bigg(\delta\Big(\sum_{\nu}(k_{z,\nu} - k_{z-\hat{\nu},\nu})\Big) \\ W_{\lambda}\Big(\underbrace{\sum_{\nu}\big(|k_{z,\nu}| + |k_{z-\hat{\nu},\nu}| + 2\,(l_{z,\nu} + l_{z-\hat{\nu},\nu})\big)}_{A_x}\Big)\Bigg)$$

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- Define generalized partition function Z_{gen}:

$$Z_{gen} = Z + \sum_{i=1}^{4} c_i \sum_{x} Z_1^i(x) + \sum_{i,j=1}^{4} c_{ij} \sum_{x,y} Z_2^{ij}(x,y) \, (+ \ldots)$$

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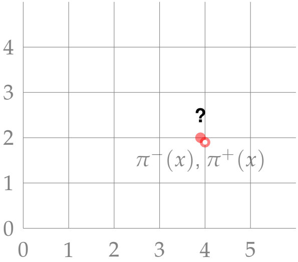

insert source/sink pair at x to obtain config. C' ?

→ Accept and set $C = C'$ with probability $P_{acc} = \min(1,r)$, where:

$$r = \frac{c_{21}}{p_s(C \to C')} \frac{W_\lambda(2 + A_x)}{W_\lambda(A_x)}$$

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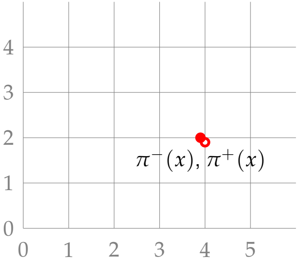
3. Lattice $\mathrm{SU}(2)$ principal chiral model at finite density

- Define generalized partition function Z_{gen}:

$$Z_{gen} = Z + \sum_{i=1}^{4} c_i \sum_{x} Z_1^i(x) + \sum_{i,j=1}^{4} c_{ij} \sum_{x,y} Z_2^{ij}(x,y) \, (+\ldots)$$

- Sample configurations for Z_{gen} with *generalized worm algorithm* . [Prokof'ev & Svistunov, 2001]

config. C contributing to Z_{gen} :



config. weight $W[C]$:

$$W[C] \propto \prod_{z} \Big(\delta\big(\overbrace{-\delta_{z,x} + \delta_{z,x}}^{0\,\forall z} + \sum_{\nu} (k_{z,\nu} - k_{z-\widehat{\nu},\nu}) \big)$$

$$W_{\lambda}\big(\underbrace{\delta_{z,x} + \delta_{z,x}}_{2\delta_{z,x}} + A_x \big) \Big)$$

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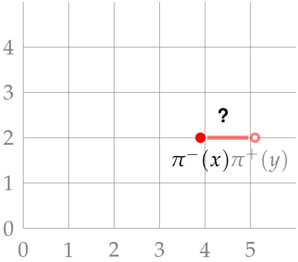

obtain new config. C' by moving worms head from x to y ?

→ Accept and set $C = C'$ with probability $P_{acc} = \min(1, r)$, where:

$$r = \frac{p_s(C' \to C)}{p_s(C \to C')} \frac{\left(\frac{\kappa}{2}\right)^{\Delta|k_{x,\nu}|} (|k_{x,\nu}| + l_{x,\nu})!}{(|k_{x,\nu}| + \Delta|k_{x,\nu}| + l_{x,\nu})!} \cdot \frac{W_\lambda(1 + \Delta|k_{x,\nu}| + A_x)}{W_\lambda(2 + A_x)} \frac{W_\lambda(1 + \Delta|k_{x,\nu}| + A_y)}{W_\lambda(A_y)}$$

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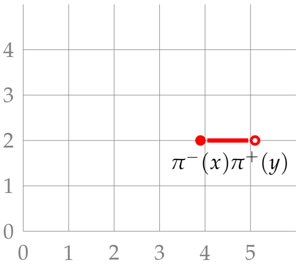

config. weight $W[C]$:

$$W[C] \propto \prod_z \Big(\delta\big(-\delta_{z,x} + \delta_{z,y} + \sum_\nu (k_{z,\nu} - k_{z-\hat{\nu},\nu}) \big) \\ W_\lambda \big(\delta_{z,x} + \delta_{z,y} + A_x \big) \Big)$$

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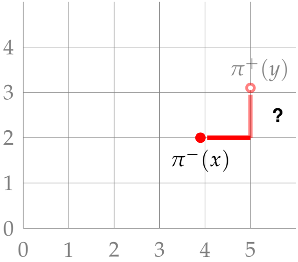

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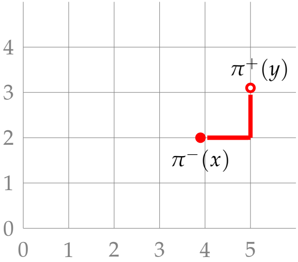

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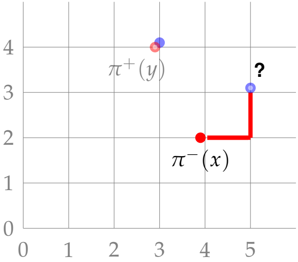

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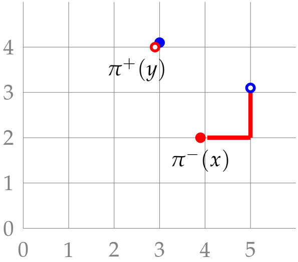

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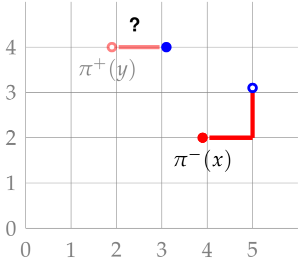

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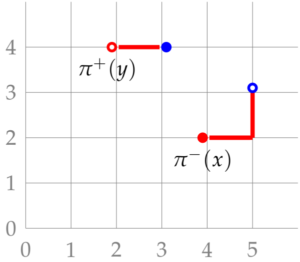

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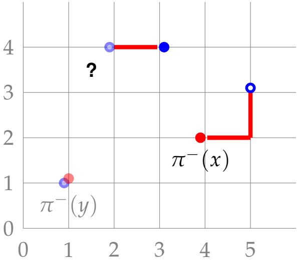

3. Lattice SU(2) principal chiral model at finite density

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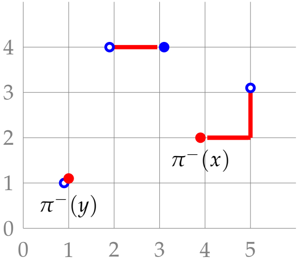

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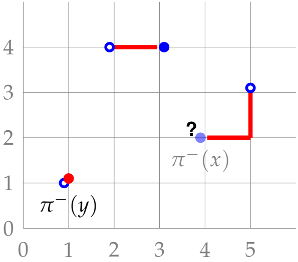

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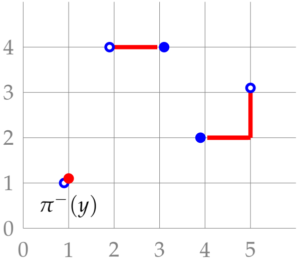

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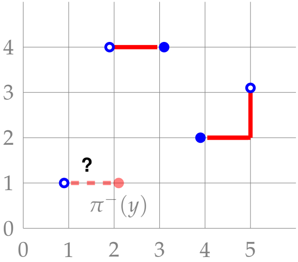

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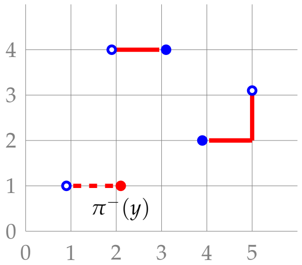

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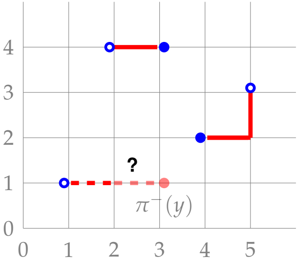

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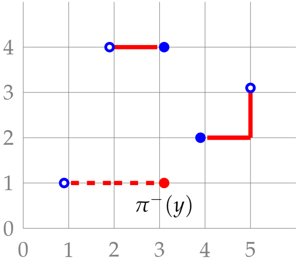

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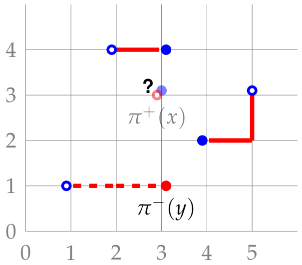

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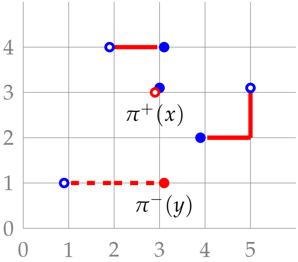

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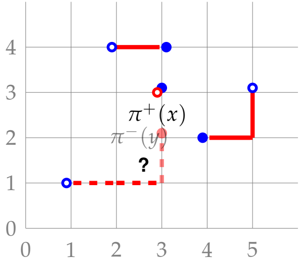

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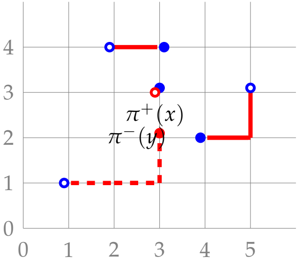

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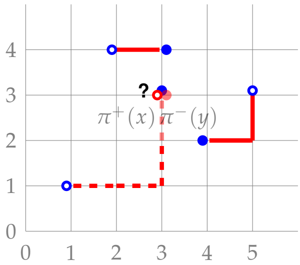

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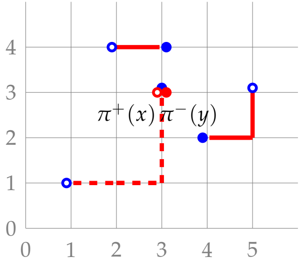

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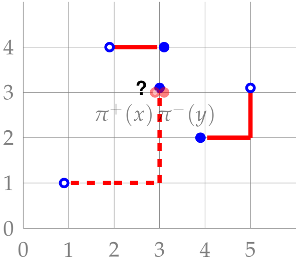

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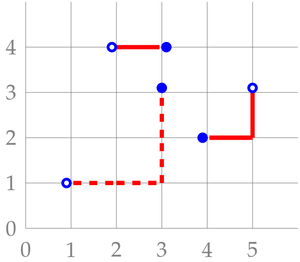

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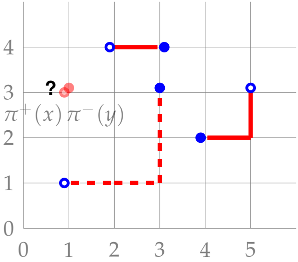

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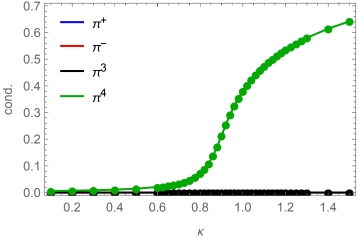
3. Lattice $SU(2)$ principal chiral model at finite density

→ Example results:

pion condensates vs. κ
($s^4 = 0.02$)



pion masses vs. κ
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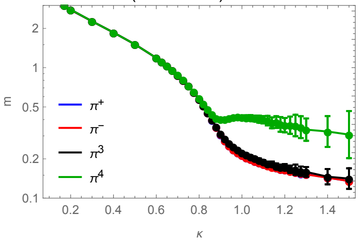
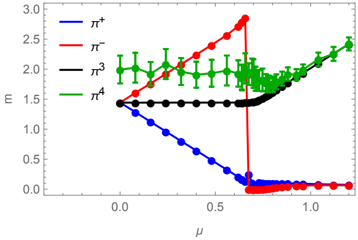


pion masses vs. μ



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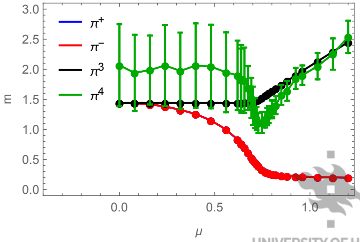

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4. Conclusions

- First principles method for QCD at finite baryon density needed to verify conjectured phase diagram. Intermediate density region inaccessible in lattice QCD due to sign problem.

- So far no reliable method to overcome sign problem advanced enough to be applicable to QCD at finite density.

- No sign problem in isospin QCD but lattice saturation and symmetry breaking can also cause problems.

- Introduction of chemical potential also leads to sign problem in bosonic models. But this can often be overcome by changing to flux variable representation.

Thank you!

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