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Signal extraction for simulated games with a large number of players

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Abstract

A signal extraction problem in simulated games is studied. A modelling technique is proposed for deriving beliefs for players in simulated games. Since standard Bayesian games provide conditions for beliefs on the basis of the common prior assumption, they do not allow for non-uniform beliefs unless the game has some dynamic structure that allows for learning. The framework presented allows for deriving beliefs by characterizing the reliability of the signals, and the players' degree of confidence in these signals. This makes it particularly suitable for games with a large number of heterogenous players.

Keywords: Signal extraction; Simulated games; Beliefs; Heterogenous players

1. Introduction

In standard Bayesian incomplete information models, the players' actions are independent of the realisations of random variables because they are assumed to know the probability distributions for the relevant random variables, but not the realisations of these variables (e.g. Harsanyi, 1967–1968, 1995). These models assume that the players start with common priors and update them with Bayes' rule as the play unfolds. Bayesian models have proven to be very useful in game theory but they are not applicable in all circumstances. For

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example, in games where the players do not have the opportunity of updating their beliefs by observing other players' choices, they may update their beliefs only if they obtain a signal that contains some valuable information for them.

A decade ago Carlsson and van Damme (1993) proposed an alternative to the Bayesian approach of analysing incomplete information in games; the *global games*. See Morris and Shin (2001) for a review of the literature on global games. Frankel et al. (2003) generalise the results of Hansson and van Damme to arbitrary numbers of players and actions.

In this paper, we will propose a modelling technique that is similar to global games and to *statistical signal extraction* in that the players are assumed to observe a *perturbed signal* of the underlying true game. However, instead of presenting an analytical model, our model is best applied in simulated games with a large number of heterogenous players.

A model where the players receive perturbed signals concerning the true game is particularly appropriate when the preference profile (the set of preferences for all players) is drawn from some symmetric distribution, and the players need to know something beyond their priors about the characteristics of a large population of heterogenous players. Such models are not to be confused with *signaling models*, where the players themselves send signals. We will study a setting where such characteristics include the *realised* distribution of players, i.e. the *number* of players with some particular type of preferences. More particularly, we will show how to derive beliefs in simulated games where the profile of player types is generated with a uniform distribution such that each player type is equally likely.

It is difficult to model the beliefs of a large number of heterogenous players because it is practically impossible to collect information on such beliefs. Our approach provides one possible way of dealing with such situations because we characterise the players' information by the *reliability of signals* they receive, and by the *degree of confidence* that they have for these signals. The framework allows us to derive beliefs for a large number of players with heterogenous preferences who receive different signals.

Since the terms 'reliability' and 'degree of confidence' have various meanings in different frameworks, let us emphasize at the outset that the reliability of the players' information is a property of the *signals* rather than an intentional state of the players in our model. It is formalised as the standard deviation of the perturbations. The degree of confidence also concerns the signals rather than the beliefs derived from them. The degree of confidence *affects* the players' beliefs, but it is conceptually and formally different from those beliefs.

One area of application for our information model is voting theory where simulations with the uniform distribution on player types is known as the *impartial culture* assumption. See Tsetlin et al. (2003) and Gehrlein (2002) for recent discussions of impartial culture. In principle, the technique is general enough to be applicable in any situation with a large number of players, but the fact that we derive beliefs for players whose types are drawn from a uniform distribution of course limits the applicability of the model.

The structure of the paper is the following. Section 2 delineates the similarities and differences of our approach to global games. Section 3 describes the signals. The beliefs are derived from these signals in Appendix A. In Section 4 we discuss how the reliability of the signals and the players' confidence in these signals affect the beliefs. Section 5 compares our concept of the degree of confidence to some previous conceptualizations.

2. Global games

A *global game* (Carlsson and van Damme, 1993) is an incomplete information game where the actual payoff structure is determined by a random draw from a given class of games and where each player makes a noisy signal of the selected game.

Consider a situation in which the players know that some game in the class of games G will be played, but they do not know which one. A class of games is a set of games with a set of players $I\,(i = 1, 2, \ldots, N)$ and a set of possible payoff profiles Π.

Initially, the players have common prior beliefs represented by a probability distribution with support on G. Before choosing an action, each player gets additional private information in the form of a perturbed signal of the actual game g to be played. The resulting incomplete information game is thus called a global game. It can be described by the following steps:

1. Nature selects a game g from G.
2. Each player observes g with some noise.
3. Players choose actions simultaneously.
4. Payoffs are determined by g and by the player's choices.

Player i:s signal is described by a random variable S_i^ε which is defined by:

$$S_i^\varepsilon = g + \varepsilon r_i,$$

where r_i is a realisation of a random variable, and ε is a scale parameter. The players are thus assumed to observe the realised game g, plus an error term εr_i. The players' signals are correlated, because they are noisy signals of the *true game*.

Our approach differs from global games as follows. First, most contributions in global games derive limit uniqueness results assuming that ε approaches zero. In this sense the signals in global games are 'close' to being correct, whereas in our approach parameter ε may in principle be of any size whatsoever. Our players are thus allowed to be 'less' informed than the players in global games.

Second, it is usually assumed in global games that as $\varepsilon \to 0$, each player becomes certain that she and her opponent have observed the true game. In contrast, we may consider parameter ε as unobservable and not necessarily known. Therefore, even if the reliability of the signals was perfect (i.e. ε was zero), we need not assume that the players are certain to have observed the true game.

Third, our model is better suited for situations with parametric rationality than strategic rationality (Elster, 1983). We derive *beliefs* for the players on the basis of what they know about the payoff profile, but we do not derive equilibrium strategies. Our model of signals and beliefs can be used *together* with different models that formulate the players' expected utilities. Since we do not present any particular application in this paper, we will not present an account for how the players can take other players' optimal strategies into account. For an example of how this can be done, see Lehtinen (2005).

3. The signals

The players' preferences are defined on a set X of items. X contains some items that the players rank or compare with each other. The most typical application of such a model is one where the players are interested in whether a majority of players prefer j to k or vice versa. The model is obviously applicable to any number of pairwise comparisons of items.

Let N denote the number of players and $n_g(j \succ k)$ the number of players who prefer j to k $(j, k \in X)$ in a simulated game g. Since all the symbols will be defined for a given simulated game g, we will not subscript our variables by 'g' in the sequel.

Let $n_i(j \succ k) = 1$, if player i prefers j to k, and $n_i(j \succ k) = 0$, if player i prefers k to j. In this simple setting, a player's *type* refers merely to whether she prefers j to k. Since we assume that the profile of player types for a simulated game g is generated with *impartial culture*, each type is equally likely. $n(j \succ k)$ can thus be viewed as a sum of N Bernoulli trials, $n(j \succ k) = \sum_{i=1}^{N} n_i(j \succ k)$, and the probability p that such a Bernoulli trial results in the outcome $n_i(j \succ k) = 1$ is $\frac{1}{2}$.

The players are assumed to obtain a perturbed signal of the number of players who prefer j to k. One way of writing the signal is as follows:

$$s_i^{\varepsilon} = \frac{n(j \succ k)}{N} + \varepsilon r_i, \tag{1}$$

where ε is a scaling factor and r_i is a realisation of a random variable P. ε reflects the *reliability of the signal*. In this paper we will assume that the variable P is standard normal; $P \sim N(0, 1)$.

What we want to do is to derive the probability that the number of players who prefer j to k is larger than $\frac{N}{2}$, given a signal S_i^{ε}. Let $p_i(j, k)$ denote such a probability for player i:

$$\begin{align} p_i(j, k) &= prob(n(j \succ k) > n(k \succ j)) \tag{2}\\ &= prob\left(n(j \succ k) > \frac{N}{2}\right) \tag{3}\\ &= prob\left(\frac{2n(j \succ k)}{n(j \succ k) + n(k \succ j)} > 1\right). \tag{4} \end{align}$$

The derivation of such probabilities requires knowledge of the variance of the variable $n(j \succ k)$. In simulated games generated with the impartial culture assumption, this variance is Np^2.

Since $n(j \succ k)$ is the sum of N Bernoulli trials, the Central Limit Theorem implies that the random variable $n(j \succ k)/N$ can be approximated with a normally distributed random variable $\mathcal{N}(j \succ k)$. Naturally, invoking the central limit theorem restricts the applicability of this model to games with a relatively large number of players.

Let us define a *perturbation* R_i as $R_i = \varepsilon r_i$. The signal can now be written as a sum of two normally distributed random variables:

$$s_i^{\varepsilon} = \mathcal{N}(j \succ k) + R_i. \tag{5}$$

Before deriving beliefs from such signals, let us point out that it will usually be more convenient to use a *standardized sum* of Bernoulli trials, $Q(j \succ k)$, instead of the variable $n(j \succ k)$ itself. The standardised sum is given by:

$$Q(j \succ k) = \frac{n(j \succ k) - Np}{\sqrt{Np^2}}. \tag{6}$$

In models with impartial culture $p = \frac{1}{2}$, so that this is

$$Q(j \succ k) = \frac{2n(j \succ k) - N}{\sqrt{N}}. \tag{7}$$

A *standardised signal* of player i is then given by

$$s_i^{\varepsilon} = \frac{2n(j \succ k) - N}{\sqrt{N}} + \varepsilon r_i \tag{8}$$

$$= Q(j \succ k) + R_i. \tag{9}$$

Deriving the beliefs from such signals involves standard statistical inference. The derivation is relegated to an appendix because it is somewhat tedious. Eq. (A.13) in the Appendix shows that the players' beliefs are given by

$$p_i(j,k) = 1 - \Phi\left(-\frac{1}{\varepsilon\sqrt{1+\varepsilon^2}} s_i^{\varepsilon}\right). \tag{10}$$

4. Reliability of signals and confidence

We will now consider how the degree of confidence and the reliability of the signals are interpreted in our model. Let us define the random variable X as follows: $X = \left(Q(j \succ k) | S = s_i^{\varepsilon}\right)$. X is the conditional value of the standardised variable $Q(j \succ k)$, given the signal s_i^{ε}. Inserting the standard deviations $\sigma_R = \varepsilon$, and $\sigma_Q = 1$ into Eq. (A.8) in the Appendix gives the density of variable X:

$$f_X(x) = \frac{\sqrt{1+\varepsilon^2}}{\sqrt{2\pi}\varepsilon} \exp\left(-\frac{1}{2}\frac{\left(1+\varepsilon^2\right)}{\varepsilon^2}\left(x - \frac{s_i^{\varepsilon}}{\left(1+\varepsilon^2\right)}\right)^2\right). \tag{11}$$

The expected value of X is thus

$$E[X] = E\left[Q(j \succ k) | S = s_i^{\varepsilon}\right] = \frac{s_i^{\varepsilon}}{\left(1+\varepsilon^2\right)}. \tag{12}$$

Eq. (12) has a natural interpretation. The smaller the variance $\left(\varepsilon^2\right)$ of the error term R_i, the more exact information the signal provides of the variable $Q(j \succ k)$, and the more it will be rational to update the beliefs.

Note that

$$\lim_{\varepsilon \to 0} E\left[Q(j \succ k)|S = s_i^\varepsilon\right] = s_i^\varepsilon. \tag{13}$$

Hence, as the error term R_i $(=\varepsilon r_i)$ approaches zero, the signal provides more and more exact information on the ratio $\frac{2n(j\succ k)}{n(j\succ k)+n(k\succ j)}$ or the corresponding standardised ratio $\frac{2n(j\succ k)-N}{\sqrt{N}}$. Furthermore,

$$\lim_{\varepsilon \to \infty} E\left[Q(j \succ k)|S = s_i^\varepsilon\right] = 0. \tag{14}$$

Hence, as the variance of the error term approaches infinity, the expected value of the conditional value of the standardised variable Q approaches zero. This means that the signals become more and more uninformative as the variance of the perturbations increases.

It may not be realistic to assume that the players know the reliability ε of their signals. In such cases the players may be assumed to formulate expectations $E_i(\varepsilon)$ concerning the reliability of their signals ε. The player's beliefs can then be derived using a modified version of Eq. (10):

$$p_i(j,k) = 1 - \Phi\left(-\frac{1}{E(\varepsilon)\sqrt{1+[E(\varepsilon)]^2}}s_i^\varepsilon\right). \tag{15}$$

Let us say that ε denotes the *reliability of signals*, and $E_i(\varepsilon)$ the *degree of confidence* in these signals. Considering Eq. (8), we may now define the following concepts:

Definition 1. The *reliability of signals*, ε, is the standard deviation of the perturbations R_i of the signals.

Definition 2. The *degree of confidence* in the signals is the expectation of the reliability of the signals $E(\varepsilon)$.

Here is how the proposed model can be used in computer simulations. We can test how the outcomes differ when we keep the preference profile fixed but vary the reliability of the signals and the degree of confidence. It is usually convenient to assume that all players and player types have the same reliability of signals and the same degree of confidence, but this is by no means necessary. It is also possible to study cases where the players are systematically over-confident ($E(\varepsilon) < \varepsilon$) or under-confident ($E(\varepsilon) > \varepsilon$).

It seems reasonable to assume that the reliability of the signals is the same for all players because the differences in the realized values of the perturbations already generate variation in the quality of the signals. However, it may also be reasonable to assume that the reliability of the signals is different for different types of players. On the other hand, making ε different by drawing it from the *same* probability *distribution* for all players would merely make the model more complicated without changing its substance. In contrast, it may be reasonable to assume that different people have different degrees of confidence in the signals even if the reliability of the signals is the same for all players. After all, if the players are assumed not to know what the reliability of the signals is, it seems natural to assume that they may have different degrees of confidence in the signals. Making this assumption makes sense if one is interested in studying how differences in the degree of confidence affect the results of the model. It may also be perfectly justified to assume the same reliability and degree of confidence for all players in a given simulation round.

Definition 3. The players have a *correct degree of confidence* if their degree of confidence in their signals equals the reliability of these signals; $E_i(\varepsilon) = \varepsilon_i$ for all $i \in I$.

The correct degree of confidence means that the players' beliefs about the quality of their signals reflects the real quality of those signals. The standard way of distinguishing between objective and subjective interpretations of probabilities is to say that probabilities can be interpreted as objective if they are based on a known probabilistic process. Probabilities are subjective if they are not based on such processes. It may thus be said that models assuming a correct degree of confidence incorporate objectively interpreted probabilities.

The smaller ε is, the more reliable a player's signals are, and the smaller $E_i(\varepsilon)$ is, the greater the player's degree of confidence in her signals. If $\varepsilon = 0$ for all $i \in I$, we say that the players have *perfectly reliable information*. However, if the players are not assumed to know the value of ε even though the players have the same signals as they would have in a corresponding complete information game, this does not yet imply that they act in the same way as players with complete information.

If $\varepsilon = E_i(\varepsilon) = 0$ for all players, i.e. if the players have both perfectly reliable information and a correct degree of confidence in their signals, the players' beliefs correspond to the knowledge of players in a corresponding complete information game. In this sense, complete information games can be viewed as a special case of our information model. To see this, note first that

$$p_i(j,k) > \tfrac{1}{2} \quad \Leftrightarrow \quad s_i^{\varepsilon} > 0,$$

$$p_i(j,k) = \tfrac{1}{2} \quad \Leftrightarrow \quad s_i^{\varepsilon} = 0,$$

$$p_i(j,k) < \tfrac{1}{2} \quad \Leftrightarrow \quad s_i^{\varepsilon} < 0.$$

With $\varepsilon = E(\varepsilon)$, and inserting (8) into (15), we have

$$\begin{aligned}
&\lim_{E(\varepsilon)\to 0} \; p_i(j,k) \\
&\quad = \lim_{E(\varepsilon)\to 0} \left\{ 1 - \Phi\left(-\frac{1}{E(\varepsilon)\sqrt{1+E(\varepsilon)^2}} \left[\frac{2n(j \succ k) - N}{\sqrt{N}} + \varepsilon r_i \right] \right) \right\} \\
&\quad = 1 \quad \Leftrightarrow \quad \frac{2n(j \succ k) - N}{\sqrt{N}} > 0 \\
&\quad = 0 \quad \Leftrightarrow \quad \frac{2n(j \succ k) - N}{\sqrt{N}} < 0.
\end{aligned}$$

Even if $\varepsilon \neq 0$, it is possible that a player's perturbed signal corresponds exactly to the true value of the variable Q, if r_i happens to be exactly zero. This is very unlikely, of course, because the perturbations are normally distributed. If $r_i = 0$, and $E_i(\varepsilon) \neq 0$, player i's signal is essentially the same as in a corresponding complete information game, but again, she will not act as if she had complete information because player i's beliefs are not degenerate (0 or 1) when she does not have full confidence in her signals. In other words, if a player happens to guess the ratio $\frac{2n(j \succ k)-N}{\sqrt{N}}$ correctly, she is not willing to act on the basis of this guess if she believes it is based on highly dubious evidence.

5. Relation to some previous literature

Many previous accounts have considered the degree of confidence *in one's beliefs*. Here, however, we model the degree of confidence *in one's signals*. This is why we need not invoke second-order beliefs (e.g., Marschak, 1975; Borch, 1975), or intervals of beliefs (e.g. Good, 1962; Gärdenfors and Sahlin, 1982) to take into account the players' confidence. These approaches suffer from well-known weaknesses. The degree of confidence in one's beliefs should already be taken into account in the first-order probabilities and thus the second-order probabilities are superfluous (see Savage, 1954, p. 58; de Finetti, 1977). If the upper and lower probabilities in the interval do not yield the same recommendations for action, there is no evident way to choose between the different actions (e.g. Skyrms, 1990, p. 113).

Second-order beliefs and intervals of beliefs have been proposed as a solution to Ellsberg's (1961) paradox. The literature that has tried to respond to Ellsberg's experiments has been concerned with two related concepts; the *degree of confidence* in one's probability judgments and the *ambiguity* of the players' information. See also the papers on the third kind of solution; non-additive probabilities (Gilboa, 1987; Schmeidler, 1989). Our approach is not designed nor suitable for modelling ambiguity because the players are always assumed to know the form of the distribution that is of interest to them. At the same time, the degree of confidence has a natural interpretation in our model.

Savage (1954, p. 68) denies that the degree of confidence in one's information can have an effect on a person's judgment of probabilities: "...the particular personalistic view sponsored here does not leave room for optimism or pessimism, however these traits be interpreted, to play any role in the person's judgment of probabilities". But since we model the degree of confidence *in the signals* rather than the degree of confidence *in the probability judgments*, we arrive at unique probabilities that may be used in standard expected utility calculations. This is why we can sidestep Skyrms' criticism even though we explicitly model the players' degree of confidence.

6. Conclusions

Our account of perturbed signals is particularly well suited for modelling situations where a large number of players have heterogenous preferences and beliefs. It is designed to be used as a part of a larger expected utility model where the differences in the players' beliefs play an important role. Since the beliefs are based on the realised profile of *player types* rather than on a profile of equilibrium *strategies*, the signal extraction belief model does not itself provide any way of taking other players' strategic behaviour into account.

Therefore, the methods for deriving the players' beliefs presented in this paper are most appropriate in situations where the players' types provide sufficiently adequate information on their actions. Given that the model is intended to be used in models with a large number of heterogenous players, and given that they have merely imprecise information on other players' types, it may well be reasonable to assume that the players do not know very much about individual players' degree of confidence. In such circumstances it is reasonable to ignore how the other players' degree of confidence might influence the degree of reliability of the information.

Voting theory provides an obvious field in which to apply our information model. There are a large number of voters with different preferences for the various candidates or alternatives. A typical piece of information that interests the voters is whether candidate A will obtain more votes than candidate B. The model presented here can be used to derive probabilities with which candidate A obtains more votes than candidate B by assuming that the probabilities are based on perturbed signals of the voters' real preferences for A and B.

There is nothing, however, in the information model that restricts its use to voting simulations. It remains to be seen whether the model can fruitfully be applied in other fields of research.

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Appendix A. Deriving the beliefs

We will now derive the belief of a player who has obtained a signal s_i^ε. In what follows in this appendix, we will omit the subscripts denoting the individuals (i), the candidates (j and k) and the simulated game (g) in order to make it easier to read the formulas. We will thus write the signal (Eq. (9)) as

$$S = Q + R. \tag{A.1}$$

R is a random variable with mean zero and variance ε^2. Since variable Q is standard normal, the signal S can be viewed as a sum of two normally distributed random variables $Q \sim N(0, 1)$ and $R \sim N(0, \varepsilon^2)$.

Let σ_Q (=1) and σ_R (=ε) denote the standard deviations of Q and R, respectively. We will now derive a conditional distribution for the variable Q, $F(Q < 0|S = s_i^\varepsilon)$.

The density of Q is

$$f_Q(q) = \frac{1}{\sqrt{2\pi}\sigma_Q} \exp\left[-\frac{1}{2}\left(\frac{q}{\sigma_Q}\right)^2\right], \tag{A.2}$$

and the density of R is

$$f_R(r) = \frac{1}{\sqrt{2\pi}\sigma_R} \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma_R}\right)^2\right]. \tag{A.3}$$

Now $q + r = s$ so that $r = s - q$. Let us now use

$$\begin{aligned} &q = x, \text{ and} \\ &r = s - x. \end{aligned} \tag{A.4}$$

Since Q and R are two independent random variables, their joint density is given by the product of their densities (e.g. Casella and Berger, 1990, p. 210)

$$\begin{aligned} f_{Q,R}(q, r) &= f_Q(q) f_R(r) \\ &= \frac{1}{2\pi} \frac{1}{\sigma_Q \sigma_R} \exp\left[-\frac{1}{2}\left(\frac{q}{\sigma_Q}\right)^2 - \frac{1}{2}\left(\frac{r}{\sigma_R}\right)^2\right]. \end{aligned} \tag{A.5}$$

Using (A.4) we get

$$= \frac{1}{2\pi} \frac{1}{\sigma_Q \sigma_R} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma_Q}\right)^2 - \frac{1}{2}\left(\frac{s - x}{\sigma_R}\right)^2\right].$$

Let $D = \left[-\frac{1}{2}\left(\frac{x}{\sigma_Q}\right)^2 - \frac{1}{2}\left(\frac{s-x}{\sigma_R}\right)^2\right]$. This can be written as follows:

$$D = -\frac{1}{2}\left[\left(\frac{1}{\sigma_R^2} + \frac{1}{\sigma_Q^2}\right) x^2 - \frac{2sx}{\sigma_R^2} + \frac{s^2}{\sigma_R^2}\right].$$

Completing the square we have

$$D = -\frac{1}{2}\left[\left(\frac{1}{\sigma_R^2} + \frac{1}{\sigma_Q^2}\right)\left(x - \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_R^2} s\right)^2 + \frac{s^2}{\sigma_R^2} - \frac{s^2}{\sigma_R^4}\right]. \tag{A.6}$$

Consider now the random variable $X = (Q|S = s)$. From (A.6) we see that $X \sim N\left(\frac{\sigma_Q^2}{\sigma_Q^2+\sigma_R^2} s, \sigma_X^2\right)$, where $\sigma_X^2 = \frac{1}{\left(\frac{1}{\sigma_R^2} + \frac{1}{\sigma_Q^2}\right)}$. The density function of X is of the following form:

$$\begin{aligned} f_X(x) = A \frac{1}{2\pi} \frac{1}{\sigma_Q \sigma_R} &\exp\left[-\frac{1}{2}\left(\frac{s^2}{\sigma_R^2} - \frac{s^2}{\sigma_R^4}\right)\right] \\ &\times \exp\left[-\frac{1}{2\sigma_X^2}\left(x - \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_R^2} s\right)^2\right]. \end{aligned} \tag{A.7}$$

Now since $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$, we have

$$A\frac{1}{2\pi}\frac{1}{\sigma_Q\sigma_R}\exp\left[-\frac{1}{2}\left(\frac{s^2}{\sigma_R^2}-\frac{s^2}{\sigma_R^4}\right)\right]$$
$$\times\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma_X^2}\left(x-\frac{\sigma_Q^2}{\sigma_Q^2+\sigma_R^2}s\right)^2\right]\mathrm{d}x = 1$$

and since

$$\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma_X^2}\left(x-\frac{\sigma_Q^2}{\sigma_Q^2+\sigma_R^2}s\right)^2\right]\mathrm{d}x = \sqrt{2\pi}\sigma_X,$$

it is easy to see that

$$A = \frac{1}{\frac{1}{2\pi}\frac{1}{\sigma_Q\sigma_R}\exp\left[-\frac{1}{2}\left(\frac{s^2}{\sigma_R^2}-\frac{s^2}{\sigma_R^4}\right)\right]\sqrt{2\pi}\sigma_X},$$

so that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{1}{2\sigma_X^2}\left(x-\frac{\sigma_Q^2}{\sigma_Q^2+\sigma_R^2}s\right)^2\right]. \tag{A.8}$$

The probability $p(Q<0|S=s)$ is given by the cumulative distribution function of X:

$$F_X(x) = \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{1}{2\sigma_X^2}\left(x-\frac{\sigma_Q^2}{\sigma_Q^2+\sigma_R^2}s\right)^2\right]\mathrm{d}x. \tag{A.9}$$

Since $\sigma_X^2 = \frac{1}{\left(\frac{1}{\sigma_R^2}+\frac{1}{\sigma_Q^2}\right)} = \frac{1}{\left(\frac{\sigma_Q^2+\sigma_R^2}{\sigma_R^2\sigma_Q^2}\right)} = \frac{\sigma_Q^2\sigma_R^2}{\sigma_Q^2+\sigma_R^2}$, $\frac{1}{\sigma_X^2} = \frac{\sigma_Q^2+\sigma_R^2}{\sigma_Q^2\sigma_R^2}$, and $\frac{1}{\sigma_X} = \sqrt{\frac{\sigma_Q^2+\sigma_R^2}{\sigma_Q^2\sigma_R^2}}$, we can write Eq. (A.9) as follows:

$$F_X(x) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sigma_Q^2+\sigma_R^2}{\sigma_Q^2\sigma_R^2}}$$
$$\times\int_{-\infty}^{0}\exp\left[-\frac{1}{2}\left(x\frac{\sqrt{\sigma_Q^2+\sigma_R^2}}{\sigma_Q\sigma_R}-\frac{1}{\sigma_Q\sigma_R}\frac{\sigma_Q^2}{\sqrt{\sigma_Q^2+\sigma_R^2}}s\right)^2\right]\mathrm{d}x. \tag{A.10}$$

We will need to make two changes of variables in order to derive a functional form that can be used in computer simulations. Let $u = \sqrt{\sigma_Q^2+\sigma_R^2}x$, so that $\mathrm{d}x = \frac{\mathrm{d}u}{\sqrt{\sigma_Q^2+\sigma_R^2}}$.

When $x=-\infty$, $u=-\infty$, and when $x=0$, $u=0$. We have thus

$$F_X(x)=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sigma_Q^2+\sigma_R^2}{\sigma_Q^2\sigma_R^2}}\frac{1}{\sqrt{\sigma_Q^2+\sigma_R^2}}$$

$$\times\int_{-\infty}^{0}\exp\left[-\frac{1}{2}\left(\frac{u}{\sigma_Q\sigma_R}-\frac{1}{\sigma_Q\sigma_R}\frac{\sigma_Q^2}{\sqrt{\sigma_Q^2+\sigma_R^2}}s\right)^2\right]\mathrm{d}u$$

$$=\frac{1}{\sqrt{2\pi}\sigma_Q\sigma_R}\int_{-\infty}^{0}\exp\left[-\frac{1}{2}\left(\frac{u}{\sigma_Q\sigma_R}-\frac{1}{\sigma_Q\sigma_R}\frac{\sigma_Q^2}{\sqrt{\sigma_Q^2+\sigma_R^2}}s\right)^2\right]\mathrm{d}u.$$

Now let $w=\frac{u}{\sigma_Q\sigma_R}-\frac{\sigma_Q}{\sigma_R}\frac{1}{\sqrt{\sigma_Q^2+\sigma_R^2}}s$, so that $\mathrm{d}u=\sigma_Q\sigma_R\mathrm{d}w$. When $u=-\infty$, $w=-\infty$, and when $u=0$, $w=-\frac{\sigma_Q}{\sigma_R}\frac{1}{\sqrt{\sigma_Q^2+\sigma_R^2}}s$, so that

$$F_X(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\sigma_Q}{\sigma_R}\frac{1}{\sqrt{\sigma_Q^2+\sigma_R^2}}s}\exp\left[-\frac{1}{2}w^2\right]\mathrm{d}w. \tag{A.11}$$

This is the cumulative distribution function of a standard normal random variable so that

$$F(Q<0|S=s)=\Phi\left(-\frac{\sigma_Q}{\sigma_R}\frac{1}{\sqrt{\sigma_Q^2+\sigma_R^2}}s\right). \tag{A.12}$$

Now since $\sigma_Q=1$, and $\sigma_R=\varepsilon$, $\frac{\sigma_Q}{\sigma_R}\frac{1}{\sqrt{\sigma_Q^2+\sigma_R^2}}s=\frac{s}{\varepsilon\sqrt{1+\varepsilon^2}}$. Player i's beliefs are given by

$$p_i(j,k)=1-\Phi\left(-\frac{1}{\varepsilon\sqrt{1+\varepsilon^2}}s_i^{\varepsilon}\right). \tag{A.13}$$

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