From infinite sum to recursive form: a simple example that helps you in the exercise 1 (Problem Set 2)

Suppose $0 < \beta < 1$, x_t stationary

$$y_t = \sum_{k=0}^{\infty} \beta^k x_{t+k} \qquad (1)$$

Shift $t \to t+1$: $y_{t+1} = \sum_{k=0}^{\infty} \beta^k x_{t+1+k}$

From (1) :

$$y_t = \beta^0 x_{t+0} + \sum_{k=0}^{\infty} \beta^{k+1} x_{t+1+k}$$

$$= x_t + \beta \underbrace{\sum_{k=0}^{\infty} \beta^k x_{t+1+k}}_{= y_{t+1}}$$

$$= x_t + \beta y_{t+1}$$

Define lag/lead operator $L x_t = x_{t-1}$ and $L^{-1} x_t = x_{t+1}$. Then

$$y_t = \sum_{k=0}^{\infty} \beta^k x_{t+k} = \sum_{k=0}^{\infty} \beta^k L^{-k} x_t$$

$$= \frac{1}{1 - \beta L^{-1}} x_t \quad \Leftrightarrow$$

$$(1 - \beta L^{-1}) y_t = x_t \quad \Leftrightarrow$$

$$y_t = \beta y_{t+1} + x_t$$