Macroeconomics 2:

Short Term Fluctuations

Antti Ripatti

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Contents

1 Outline

Goals

Sources of business cycles
Measurement

How households and firms respond to shocks
Consumption, investments (and if time allows, foreign trade)

What is the policy response
What should (monetary and fiscal) policy to do?

What is special in current economic situation
Financial market imperfections, Zero lower bound, multiple steady-states

Wish list

1. Introduction: What are business cycles
2. Consumption
3. Labour supply
4. Investments
5. Real business cycles
6. Fiscal policy
7. Imperfect competition and price rigidities
8. Monetary policy
9. Open-economy issues
10. ZLB, fiscal policy, QE, . . .

2 Textbook view

2.1 Real rates and inflation

Fisher equation
Fisher equation

$$\underbrace{i}_{\text{nominal interest rate}} = \underbrace{r}_{\text{real interest rate}} + \underbrace{\pi}_{\text{inflation}} .$$

Assume *sticky prices*, then

$$r = i - \pi \tag{2.1}$$

real rate may be controlled if one is able to control the nominal interest rate.

Central bank behaviour

Central bank targets inflation rate $\pi^\star$ and responds to vigorously to inflation deviating from the target

$$i = r^\star + b(\pi - \pi^\star),$$

where $r^\star$ is the *natural real rate* (determined exogenously), and $b > 1$ ("vigorous").

Combine this with the real rate equation (2.1) to obtain

$$r = r^\star + b(\pi - \pi^\star) - \pi = r^\star + \underbrace{(b-1)}_{>0}\pi - b\pi^\star$$

higher inflation leads to higher real rate.

2.2 Aggregate demand

Aggregate demand

Consider aggregate demand (AD) curve in output Y and inflation π coordinates:

$$Y(\pi) = C(\pi) + I(\pi) + G$$

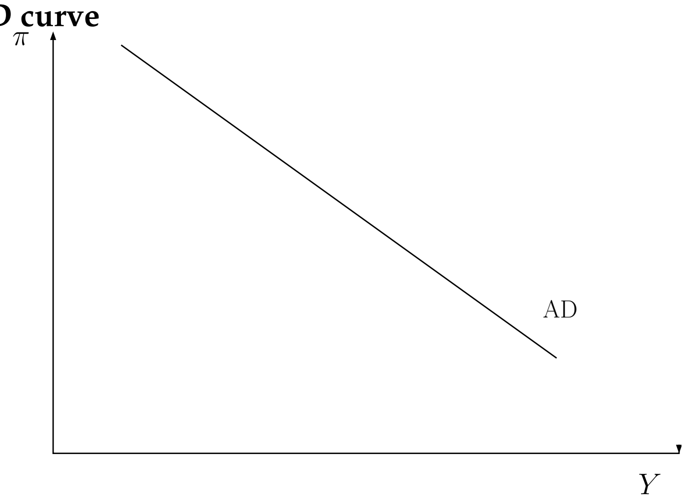
Since consumption C and investments I are inversely related to real rate, the output Y will be downward-sloping, ie

$$\text{increasing } \pi \longrightarrow \text{ increasing } r \longrightarrow \text{ decreasing } Y$$

Real rate and components of aggregate demand

The essential point of this course is to provide rigorous story how consumption and investments depend on the real rate (and other factors).

AD curve



2.3 Aggregate supply

Long-run aggregate supply
Long-run aggregate supply (LRAS) depends on

- production technology
- amount of factors of production

but not on inflation/price level $\longrightarrow$ it is vertical in the (π, Y) coordinates.

Sometimes this level is called *natural level of output*. Can be time-varying!

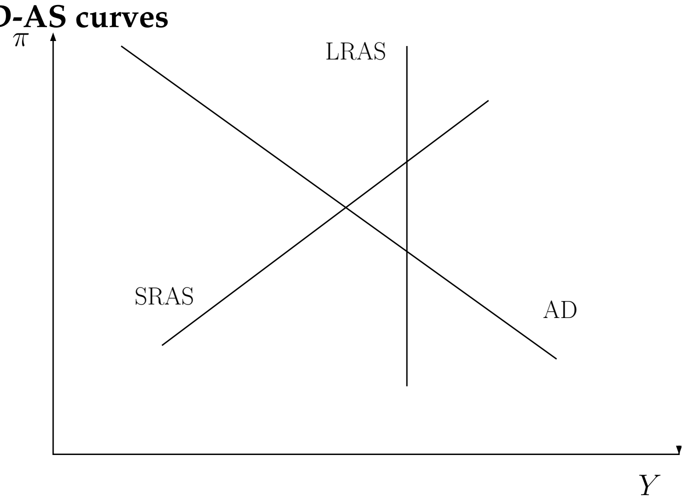
Short-run aggregate supply
Short-run aggregate supply (SRAS) is upward-sloping, ie not vertical, due to

- expectation errors
- sticky wages
- sticky prices.

(Among many other stories)

Supply side
The essential point of this course is to provide a story what drives fluctuations in the LRAS and why SRAS is upward-sloping.

AD-AS curves



Shock to aggregate demand

This course

The nominal interest rate is a "price" of a loan contract from today to a date in the future

$\longrightarrow$ *expected* inflation matters (instead of current inflation)

$\longrightarrow$ we need to be precise on the expectation formation, and intertemporal issues! Hence, we will revisit in the Fisher equation.

Since production possibilities (LRAS in the principles view) depend on the amount of factors of production, we have to be careful with the

- labour supply
- capital formation (investments)

This course. . .

To know the shape ("slope") of the SRAS, ie trade-off between inflation and output, we need to have a coherent story on this trade-off.

Given the nuts-and-bolts of a stylized economy we may study monetary and fiscal policy.

If time allows, we will study the refinements on how the interest rates and loan contracts faced by households and firms are formed: financial markets!

2.4 Phillips curve

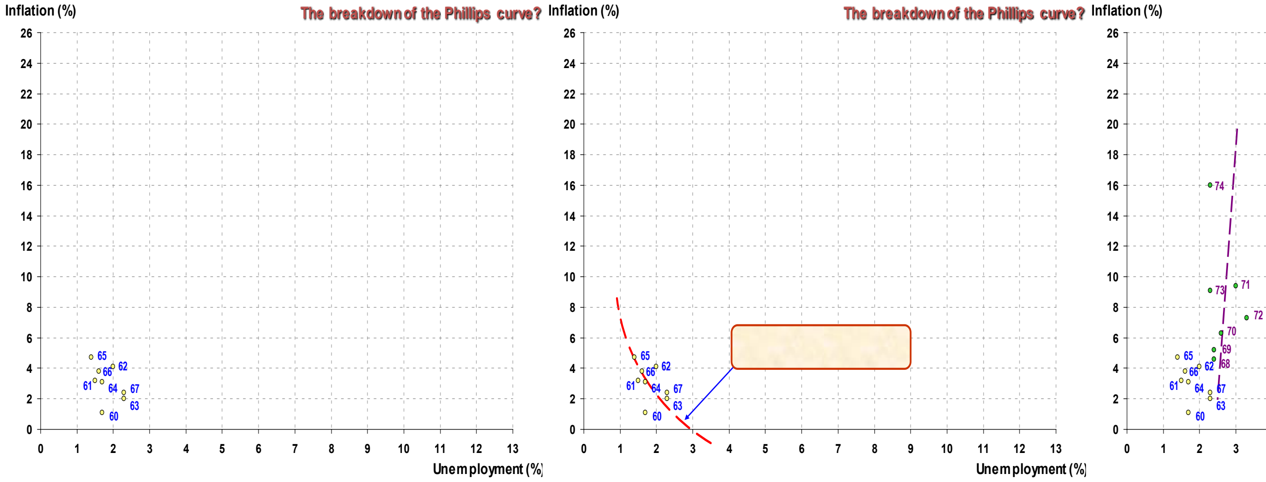
Aggregate supply

Short-run aggregate supply is often called *Phillips curve (PC)* (due to its inventor). It is often expressed in the form (π,unemployment) coordinates.

$\longrightarrow$ there is a trade-off between inflation and output (employment)

LRAS is vertical $\longrightarrow$ *no* trade-off.

History of Phillips curve



Shifting Phillips curve drove macrotheory

- 1960s demand management policies relied on the PC.
- Friedman (1968) and Phelps (1968): no long-run trade-off

  This is why we separate short and long run.

$\longrightarrow$ new theories about what shifts PC:

- Microfoundations: behavioural equations based on preferences and technologies, agents optimize
- Rational expectations: agents make no systematic errors.

2.5 Measuring business cycles

Ideas

Divide the real variables into two components

1. Long term component: moves slowly, smooth, driven by economic growth, structural changes, etc.
2. Business-cycle component: moves more quickly, cycle length of $2-8$ years.

Hodrick-Prescott Filter

- The Hodrick-Prescott filtering is probably the most commonly used method of extracting business cycle components in macroeconomics.
- The general idea is to compute the growth (trend) component g_t and cyclical component c_t of y_t by minimizing the magnitude

$$\sum_{t=1}^{T} \underbrace{(y_t - g_t)}_{c_t}^2 + \lambda \sum_{t=1}^{T-1} \left[(g_{t+1} - g_t) - (g_t - g_{t-1})\right]^2 .$$

  - The growth component g_t should not be to far from actual data y_t, ie

$$y_t - g_t$$

    should not be too high
  - The growth rate of growth component

$$(g_{t+1} - g_t) - (g_t - g_{t-1})$$

    should not fluctuate too much.
  - The smoothing parameter λ tells how much (relative) weight is given to the second objective.
    * If $\lambda = 0$, $g_t = y_t$ (no smoothing).
    * The greater λ is, the smoother the growth component. When $\lambda \to \infty$, g_t is a straight line.
    * There is a trade-off between these two goals.
    * Typically $\lambda = 1600$ for quarterly data and $\lambda = 100$ for annual data to extract the growth component whose wavelength is larger that eight years.

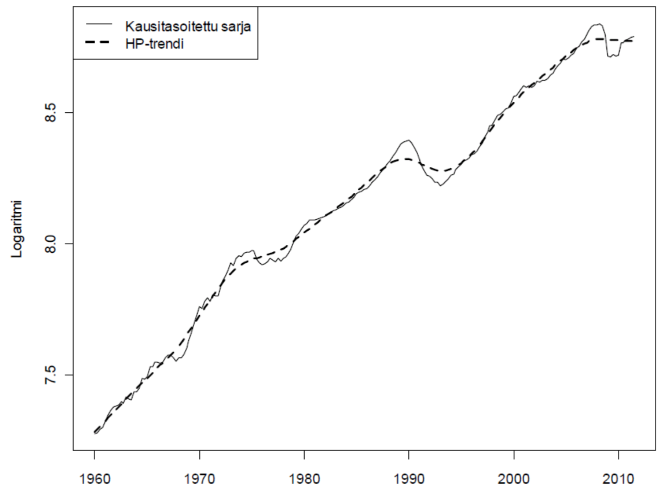
- Using the lag operator L, the cyclical component produced by the Hodrick-Prescott filter can be written as (see, Baxter and King, 1999)

$$c_t = \left[\frac{\lambda (1-L)^2 \left(1-L^{-1}\right)^2}{1+\lambda (1-L)^2 \left(1-L^{-1}\right)^2}\right] y_t.$$

  - The filter removes unit root components.
  - The filter is a symmetric infinite-order two-sided moving average.

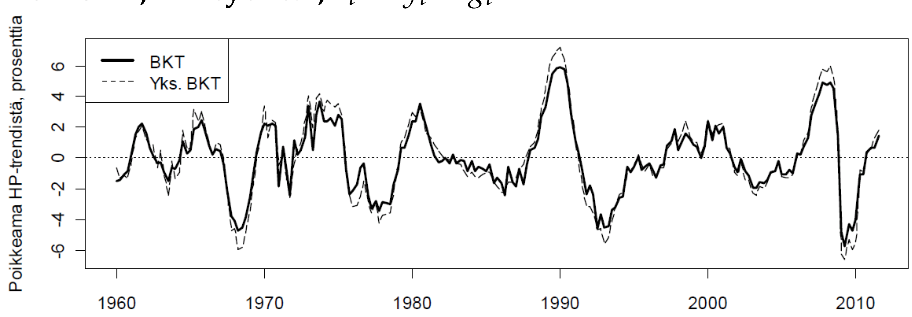
* c_t depends on past and future values of y_t, $t = 1, 2, ..., T$. Therefore, the first and last values of c_t are inaccurate and may change considerably as new observations become available.
* The filter introduces no phase shift.

Finnish GDP, y_t and HP trend g_t



Source: Ahola (2012)

Finnish GDP, HP cyclical, $c_t = y_t - g_t$



Source: Ahola (2012)

Business cycle statistics

Other statistics (aka "data moments")

- Variances; relative to output
- Autocorrelations: how consecutive observations are correlated

- Cross-correlations (a) how different variables are correlated; (b) how leads/lags of different variables are correlated
- Spectrum: how important are cycles of different frequencies
- Great ratios: consumption/output, investments/output, output/capital, labour share, ...
- Impulse responses of structural VARs.

Pitfalls
Problems with HP

- Trend and cycle are *independent*
- Each variables has its own trend; some theories say that it should be common
- It is moving-average: initial and end-point problems (observations lost)
- Passes very short term fluctuations (use bandpass filter instead)

Data moments have problem in being linear!

3 Consumption

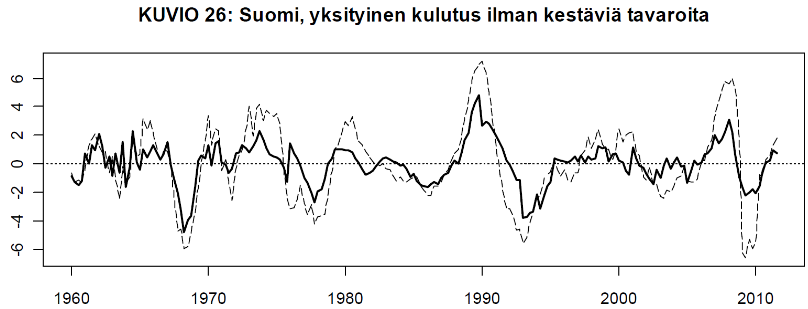
3.1 Introduction

Preliminaries
(Private/household) consumption is 60-70 % of GDP! By far the largest component.
$\longrightarrow$understanding consumption is essential in understanding the fluctuations in output
Relationship between consumption C and income Y.

- Decision on how much to consume and how much to save
- Households live more than one period.
  $\longrightarrow$They make intertemporal (across time) decisions
  $\longrightarrow$they make decisions with some view about the future
- Some people end up being creditors, and some debtors in different points of their *life-cycle*

- "saving for the rainy day", "make hey while sun shines"
- Income is mostly labour income
  ⟶consumption-leisure decision

Output and consumption (non-durables)



- substantially less volatile than output(income): *consumption smoothing*
- procyclical (goes hand-in-hand with output/income)

Problems in the textbook model

The bastard Keynesian view

$$C = c \times Y,$$

where $c \in (0, 1)$ is the marginal propensity to consume.
⟶All consumers are saving

Empirical problems:

- regressing trend consumption on the trend income results $c \approx 1$
- regressing cyclical consumption on the cyclical output results $c \ll 1$

Contradiction!

Other approaches

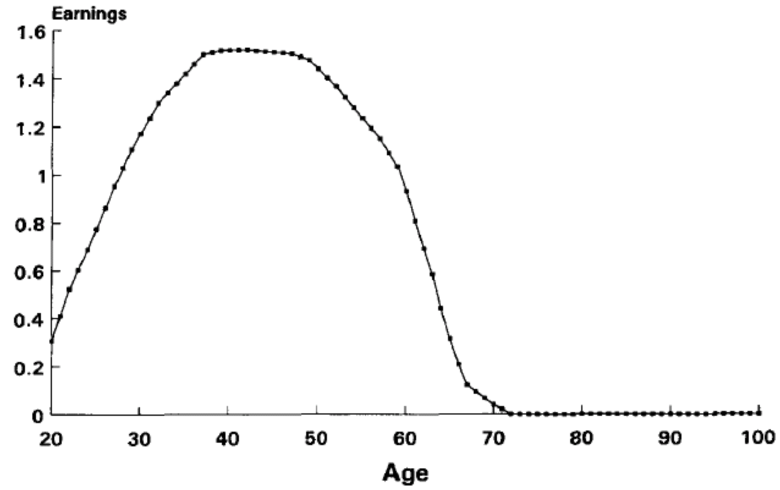
- Life-cycle theory (Ando-Modigliani-Brumberg, Tobin, etc., Ramsey-Keynes): Consumers want to smooth the consumption path, income varies along the life-cycle.
- Permament income theory (Friedman): Incomes at any date subject to random shocks, smoothing motive applies.

- Relative income hypothesis (Duesenberry): People are concerned with their consumption relative to the others' consumption (in addition to their own consumption).

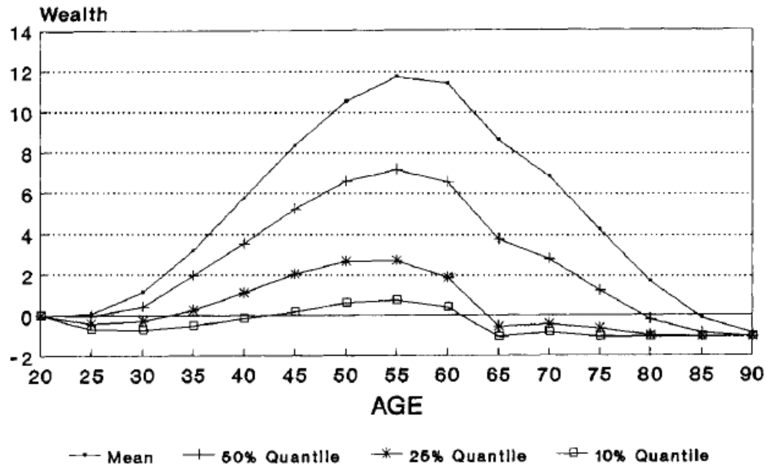
Life-cycle and permanent income theories the foundation of modern macro models (they can be integrated) but with some modifications.

Need to recall the relative income theory? Behavioral economics point of view.

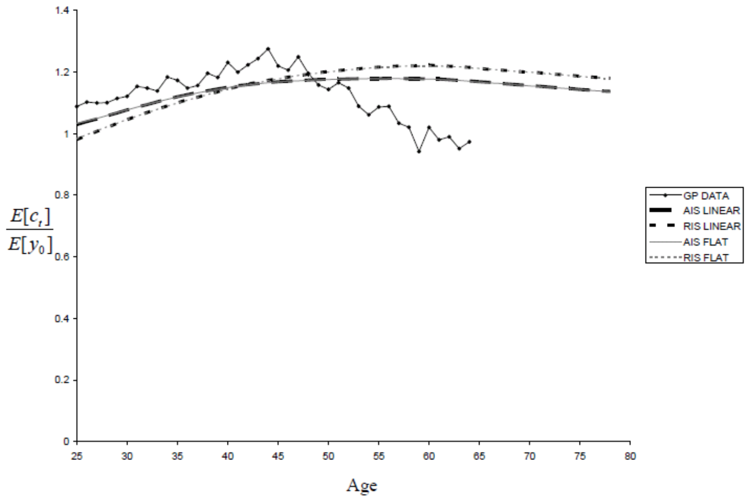
Life-cycle income



Life-cycle wealth



Life-cycle consumption



3.2 Two-period model of consumption

3.2.1 Preliminaries

Notation

Household lives two periods: t "today", $t+1$ "tomorrow" (or young and old).

Notation:

c_t consumption today

c_{t+1} consumption tomorrow

w_t income today

w_{t+1} income tomorrow

r_t real interest rate for credit/debt from today to tomorrow

s_t savings today

Households supply labour of one unit in each period.

Utility

Households utility is given by

$$U = U(c_t, c_{t+1})$$

with the typical assumptions

- $U_{c_i} \equiv \partial U/\partial c_i > 0 \; i \in \{t, t+1\}$. Utility is increasing.
- $U'_{c_i,c_j} \equiv \partial^2 U/\partial c_i \partial c_j < 0, i, j \in \{t, t+1\}$. Diminishing returns. Marginal utility is declining in total consumption $\longrightarrow$ smooths the consumption.
- Inada conditions $\lim_{c\to 0} U'(c) = \infty$ and $\lim_{c\to\infty} U'(c) = 0$

Replace today by "hot-dogs" and tomorrow by "hamburgers", and r_t with their relative price and you may use the tools learned in intermediate microeconomics.

Household does not care about the leisure. Supply fixed, 1, amount of labour. (We will generalize this later.)

(Show U and U'. Explain consumption smoothing.)

Timing of decisions

Today		Tomorrow	
w_t	c_t	w_{t+1}	c_{t+1}
$s_t = w_t - c_t$			$(1+r_t)s_t$

Budget constraint

Household earns today income $w_t \times 1$

And it consumes amount c_t, and saves the rest s_t.

The period 1 budget constraint is given by

$$c_t + s_t = w_t$$

In the next period (tomorrow) it consumes her 2nd period income $w_{t+1} \times 1$ and the savings (plus interest) from previous period:

$$c_{t+1} = w_{t+1} + (1+r_t)s_t$$

Note that she does not leave any bequests, ie is buried with empty pockets. Hence, no savings in period 2.

These budget constraints are also called *flow budget constraints*.

Note that capital market is assumed to be perfect: household can borrow or invest any amount of $s_t \in \mathbb{R}$ within the limits of the budget constraints. If $s_t > 0$ she is saving, $s_t < 0$ she is borrowing.

Intertemporal budget constraint

Solve the s_t from the second period budget constraing

$$s_t = \frac{c_{t+1}}{1+r_t} - \frac{w_{t+1}}{1+r_t}$$

and substitute it into the first period budget constraint to obtain *intertemporal budget constraint*

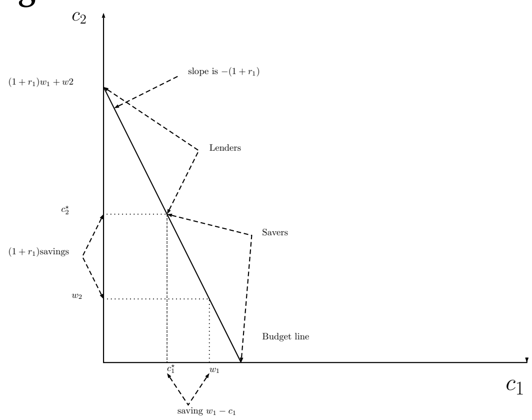
$$\underbrace{c_t + \frac{c_{t+1}}{1+r_t}}_{\text{Present value of consumption}} = \underbrace{w_t + \frac{w_{t+1}}{1+r_t}}_{\text{Present value of income = wealth, } \equiv W_t} .$$

This implies that *consumption function* is of the form

$$c_t = c_t\left(\frac{1}{1+r_t}, W_t\right)$$
$$c_{t+1} = c_{t+1}\left(\frac{1}{1+r_t}, W_t\right)$$

- consumption in each period depends on the real interest rate r_t, and life-cycle wealth W_t
  $\longrightarrow$*the time-path of the income does not matter*!
- consumption smoothing
- consumption level is not the same in each period: it depends on the time preference and interest rate.

Budget line



We want to describe the household's decision in (c_t, c_{t+1}) coordinates.

Get the budget line from the intertemporal budget constraint by expressing c_{t+1} with other variables:

$$c_{t+1} = -(1+r_t)c_t + (1+r_t)w_t + w_{t+1}$$

Consumption function

Households' control variables are c_t and c_{t+1} (they choose them!).

They need to be chosen so that they satisfy the budget constraint
$\longrightarrow$They cannot be chosen independently each other.

Wages and interest rates are exogenously given: they do not respond to household's choices. (They are economy-wide and household is small.)

The maximization problem is the following

$$\max_{c_t, c_{t+1}} U = U(c_t, c_{t+1}) \tag{3.1}$$

subject to the intertemporal budget constraint

$$c_t + \frac{c_{t+1}}{1+r_t} = w_t + \frac{w_{t+1}}{1+r_t}. \tag{3.2}$$

This should be easy to solve given the tools learned at the bachelor level!

The Lagrangean

$$\mathcal{L} = U(c_t, c_{t+1}) + \lambda \left(w_t + \frac{w_{t+1}}{1+r_t} - c_t - \frac{c_{t+1}}{1+r_t} \right).$$

Now we have an additional variable, the Lagrange multiplier λ to be determined.

The first-order conditions are

$$\begin{aligned} \mathcal{L}_{c_t} &= U_{c_t}(c_t, c_{t+1}) - \lambda = 0 \\ \mathcal{L}_{c_{t+1}} &= U_{c_{t+1}}(c_t, c_{t+1}) - \lambda \frac{1}{1+r_t} = 0 \\ \mathcal{L}_{\lambda} &= w_t + \frac{w_{t+1}}{1+r_t} - c_t - \frac{c_{t+1}}{1+r_t} = 0. \end{aligned}$$

Combining the first two gives

$$U_{c_t}(c_t, c_{t+1}) = (1+r_t)U_{c_{t+1}}(c_t, c_{t+1}) \quad (= \lambda).$$

It says that an optimizing consumer sets the marginal utility loss (LHS) of saving one consumption unit of saving one consumption unit for tomorrow equatio to the gain tomorrow in consumption terms, the return on the savings $(1+r_t)$ times the marginal utility of each unit tomorrow:

$U_{c_t}(c_t, c_{t+1})$	$=$	$U_{c_{t+1}}(c_t, c_{t+1})$	$\times(1+r_t)$
Utility lost		Utility increase	return on savings
if you save		next period per	by how many units
"one" more unit		unit of increase in c_{t+1}	next period's c can increase

Along the optimal consumption path

$$\frac{U_{c_t}(c_t, c_{t+1})}{U_{c_{t+1}}(c_t, c_{t+1})} = 1 + r_t \tag{3.3}$$

the marginal utility of consumption today relative to marginal utility of consumption tomorrow equals the "relative price" of consumption today and tomorrow $1 + r_t$. To see this, write

$$\frac{U_{c_{t+1}}(c_t, c_{t+1})}{U_{c_t}(c_t, c_{t+1})} = \frac{\frac{1}{1+r_t}}{1},$$

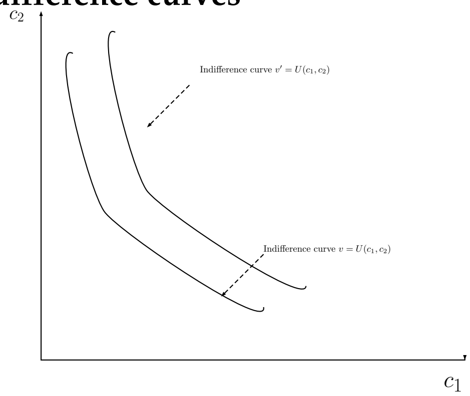
where 1 is price of consumption today and $1/(1+r_t)$ is the price of consumption tomorrow.

Note

- Eq (3.3) is *not a solution*! (Does not tell how c_t and c_{t+1} depend on exogenous variables: wages and interest rate.)
- It is an expression resulting from the optimality conditions.
- Necessary condition
- LHS is called *marginal rate of substitution* (MRS)

  Example: call c_t "hamburger" and c_{t+1} "hot dog". Then (3.3) tells that the marginal rate of subsitution between "hamburger" and "hot dog" equals to their relative price $1 + r_t$. It gives the *increase* in consumption of "hot dogs" that is required to keep the utility level constant when amount of "hamburgers" is decreased marginally.
- MRS

Indifference curves

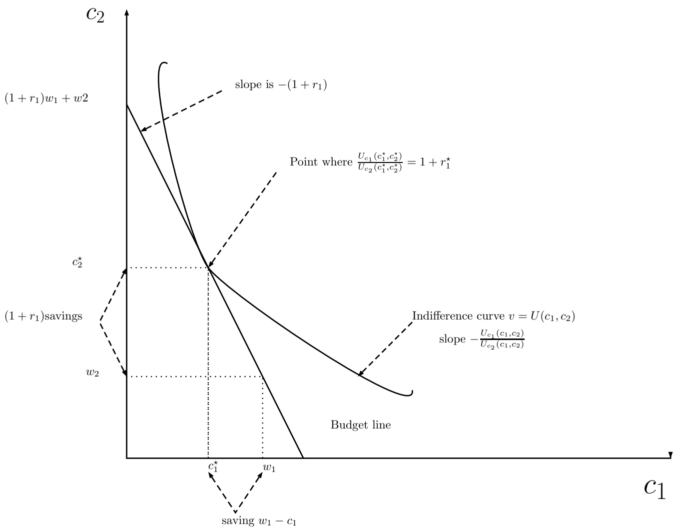


indifference curves from

$$v = U(c_t, c_{t+1})$$

by varying v appropriately.

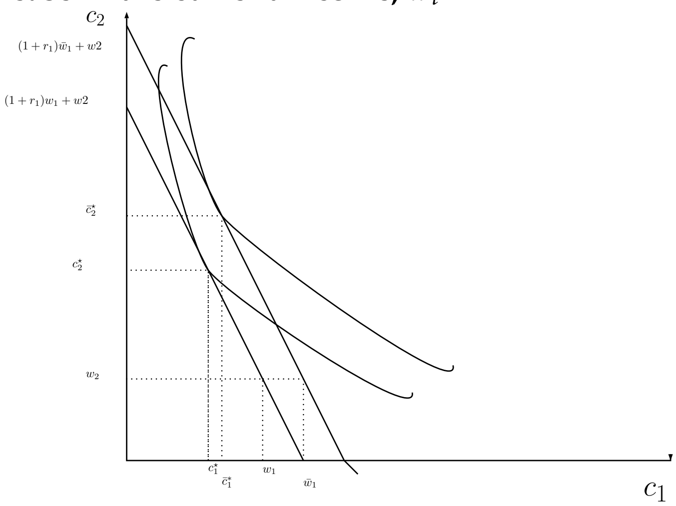
Figure



Applications

1. Increase in the current income w_t
2. Increase in the future income w_{t+1}
3. Increase in the interest rate r_t.

Increase in the current income, w_t



Consumption smoothing

Income increases from w_t to $\bar{w}_t$.

Slope of the budget constraint, $-(1 + r_t)$, does not change!

Since current and future consumption are normal goods, both will increase. Also savings increase!

Consumption smoothing:

- Consumers want to spread an increase in income over several periods. This is called *consumption smoothing*.

$\longrightarrow$ consumption is less volatile than income!

- Consistent with the evidence in the beginning of this section.
- But not quantitatively: consumption is not as smooth as theory predicts
  1. Imperfections in the credit market
  2. If all smooth, the market price will change: general equilibrium effect.

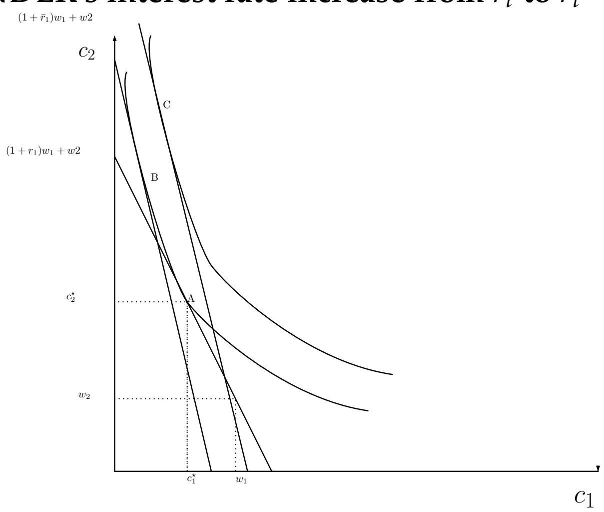
Increase in the future income w_{t+1}

Budget line shift is similar. Both c_t and c_{t+1} will increase but s_t decrease!

Temporary vs. permament changes in income

- Permanent income hypothesis (PIH) tells that the main determinant of consumption is the *permanent income*, in our setup the life-cycle income.
- Since temporary change in income has a small impact on life-cycle income, it has small impact on consumption.
- Since permanent change in income has a large impact on life-cycle income, it has a large impact on consumption.
- In our model the permanent change in income would be an increase in *both* w_t *and* w_{t+1}.

LENDER's interest rate increase from r_t to $\bar{r}_t$



Income and substitution effect

Substitution effect: A → B
$\bar{r}_t > r_t$, the slope will be higher (tilts budget line steeper). The new point will be *B* where this new slope equals the slope of the indifference curve.
$\longrightarrow c_t$ decreases, c_{t+1} increases, s_t increases.

Income effect: B → C
Intercept increases too: $(1+\bar{r}_t)w_t + w_{t+1} > (1+r_t)w_t + w_{t+1}$. Consumer is *richer*
$\longrightarrow c_t$ increases, c_{t+1} increases, s_t decreases.

Effect	c_t	c_{t+1}	s_t
Substitution	$-$	$+$	$+$
Income	$+$	$+$	$-$
TOTAL	?	$+$	?

BORROWER's interest rate increase from r_t to $\bar{r}_t$

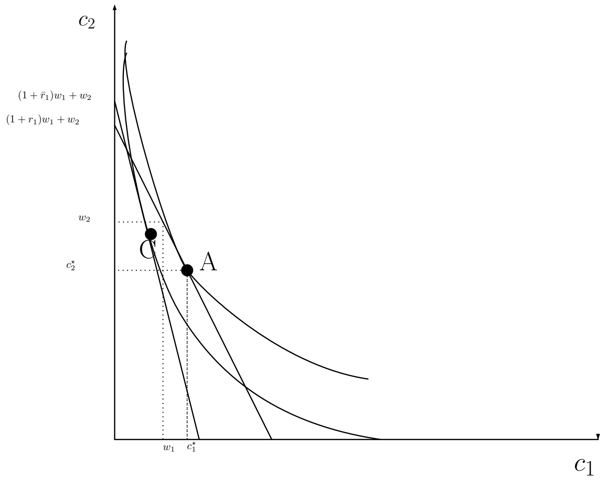
The rise in the real interest rate will make the slope of the budget line steeper but it will also make borrower poorer. This means that the budget line moves left. She moves from the point A to the point C

Relative price of future consumption is lower.

$\longrightarrow$Substitution effect: c_t decreases, c_{t+1} increases, s_t increases

$\longrightarrow$Income effect: consumer is poorer; c_t decreases, c_{t+1} decreases, s_t increases

BORROWER's interest rate increase from r_t to $\bar{r}_t$



Aggregate effect

For lenders

Effect	c_t	c_{t+1}	s_t
Substitution	$-$	$+$	$+$
Income	$+$	$+$	$-$
TOTAL	?	$+$	?

For borrowers

Effect	c_t	c_{t+1}	s_t
Substitution	$-$	$+$	$+$
Income	$-$	$-$	$+$
TOTAL	$-$	?	$+$

Aggregate effect depends on

- The relative size of income and substitution effects
- The number of borrowers and lenders.

Example: time-separable CES utility

Let's parametrize the utility function as follows

$$U(c_t, c_{t+1}) = \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma}}{1-\sigma} \quad \sigma \geq 0,\ \sigma \neq 1.$$

Marginal utilities

$$U_{c_t}(c_t, c_{t+1}) = c_t^{-\sigma}$$
$$U_{c_{t+1}}(c_t, c_{t+1}) = \beta c_{t+1}^{-\sigma}.$$

and the optimality condition:

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1 + r_t,$$

where the LHS is the MRS!

Intertemporal elasticity of substitution, time preference rate

Recall the elasticity of substitution: *increase* in relative consumption c_i/c_j when the relative price p_i/p_j *decreases* (identically p_j/p_i increases). Formally

$$\epsilon_{ij} = \frac{d(c_i/c_j)}{d(p_i/p_j)} \frac{p_i/p_j}{c_i/c_j} = \frac{\frac{d(c_i/c_j)}{c_i/c_j}}{\frac{d(p_i/p_j)}{p_i/p_j}} = \frac{d\log(c_i/c_j)}{d\log(u_{c_j}/u_{c_i})}$$

In our setting, the object of interest is relative consumption of tomorrow and today: c_{t+1}/c_t and the relative price $1/(1+r_t) = 1/MRS$.

The above elasticity may be written in the form

$$\epsilon_{1,2} = \frac{d\log(c_{t+1}/c_t)}{d\log\left(U_{c_t}(c_t, c_{t+1})/U_{c_{t+1}}(c_t, c_{t+1})\right)}$$

and in the case of CES utility it results (proof. homework):

$$\epsilon_{1,2} = \frac{1}{\sigma}$$

This is where the name CES (constant elasticity of substitution) comes from: the intertemporal elasticity of substitution is constant $1/\sigma$.

Time preference rate is the rate at which future instantaneous utilities are discounted. Imagine a discounted income stream:

$$x_0 + \frac{1}{1+r}x_1 + \left(\frac{1}{1+r}\right)^2 x_2 + \cdots ,$$

where the discounting is done with the interest rate r.

Replace the interest rate with ρ, where ρ would be the *time preference rate*. In our previous example the utility was

$$U_0 = \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma}}{1-\sigma}$$

where

$$\beta \equiv \frac{1}{1+\rho}, \quad \rho > 0.$$

Optimal consumption in the CES case

$$\max_{c_t, c_{t+1}} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma}}{1-\sigma}$$

subject to the intertemporal budget constraint

$$c_t + \frac{c_{t+1}}{1+r_t} = w_t + \frac{w_{t+1}}{1+r_t}.$$

The Lagrangian:

$$\mathcal{L} = \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma}}{1-\sigma} + \lambda \left(w_t + \frac{w_{t+1}}{1+r_t} - c_t - \frac{c_{t+1}}{1+r_t} \right).$$

Optimality (=first order) conditions:

$$\begin{aligned}
\mathcal{L}_{c_t} &= c_t^{-\sigma} - \lambda = 0 \\
\mathcal{L}_{c_{t+1}} &= \beta c_{t+1}^{-\sigma} - \lambda \frac{1}{1+r_t} = 0 \\
\mathcal{L}_{\lambda} &= w_t + \frac{w_{t+1}}{1+r_t} - c_t - \frac{c_{t+1}}{1+r_t} = 0.
\end{aligned}$$

Dividing the first two optimality conditions give

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1 + r_t$$

or

$$c_{t+1} = [\beta(1+r_t)]^{\frac{1}{\sigma}} c_t. \tag{3.4}$$

This corresponds the optimality rule that we derived above.

And substitute it into the budget constraint:

$$c_t + \frac{c_{t+1}}{1+r_t} = \underbrace{w_t + \frac{w_{t+1}}{1+r_t}}_{\equiv W_t}$$

to obtain

$$c_t + \frac{[\beta(1+r_t)]^{\frac{1}{\sigma}}}{1+r_t} c_t = W_t$$

and solve c_t

$$c_t = \left[1 + (1+r_t)^{1/\sigma - 1} \beta^{1/\sigma}\right]^{-1} W_t \tag{3.5}$$

and

$$c_{t+1} = \left[\beta^{-1/\sigma}(1+r_t)^{-1/\sigma} + (1+r_t)^{-1}\right]^{-1} W_t \tag{3.6}$$

The expressions (3.5) and (3.6) are called closed-form solution since they express the endogenous variables c_t and c_{t+1} as a function of only exogenous variables.

Consider special case: $\sigma = 1$. The above solutions simplify as follows:

$$c_t = \frac{1}{1+\beta} W_t$$

and

$$c_{t+1} = \frac{\beta}{1+\beta}(1+r_t) W_t$$

It is easy to see that it is not optimal to consume the life time wealth half and half (since $0 < \beta < 1$).

The coefficient in front of wealth is called *marginal propensity to consume out of wealth* (MPCW).

Example: Suppose agent lives two "years" and the rate of time preferences is 3 %.

- Then $\beta = 1/(1+0.03) = 0.97$, household-consumer appreciates current consumption more than future consumption and
- MPCW for the first period consumption is $1/(1+\beta) = 0.51$
- and for the second period $0.49(1+r_t)$

Note that the first period MPCW is independent of r_t only in the special case of $\sigma = 1$.

Does consumption grow over time?

- The condition for $c_{t+1} > c_t$ (using (3.5) and (3.6) with the assumption $\sigma = 1$).

$$
\begin{aligned}
& c_{t+1} > c_t \quad \leftrightarrow \\
& \frac{\beta}{1+\beta}(1+r_t)W_t > \frac{1}{1+\beta}W_t \\
& (1+r_t) > \frac{1}{\beta} \\
& r_t > \frac{1}{\beta} - 1 = \frac{1}{\frac{1}{1+\rho}} - 1 = \rho.
\end{aligned}
$$

- consumption increases if the real interest rate is higher than the time preference rate.
- although we calculated this for a very simple utility function, it also holds in more general cases.

3.3 Applications and extensions

3.3.1 Borrowing constraints

Borrowing constraints

We have assumed that a household may borrow freely at interest rate r_t.

Next we assume that the household cannot obtain loan, ie it cannot borrow:

$$s_t \geq 0.$$

Let $(c_t^\star, c_{t+1}^\star, s_t^\star)$ denote the *optimal consumption choice* in the absense *of the borrowing constraint.*

Two cases emerge:

1. If the optimal unconstraint choice satisfies $s_t^\star \geq 0$, it will still be the optimal choice.

2. If the optimal unconstraint choice satisfies $s_t^\star < 0$ (he would like to borrow), then the best she can do is the following

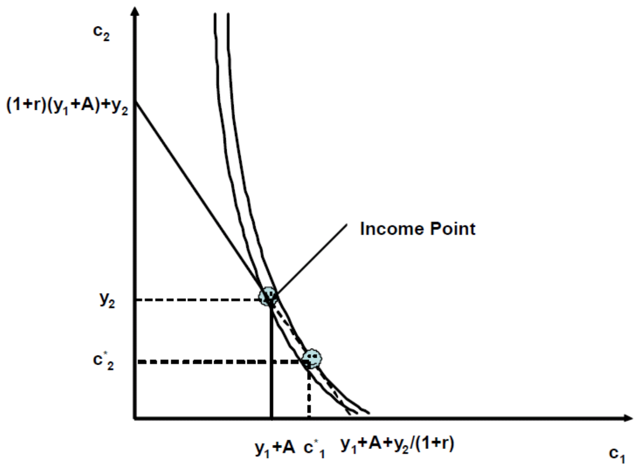
$$c_t = w_t$$
$$c_{t+1} = w_{t+1}$$
$$s_t = 0.$$

She would like to have larger c_t but cannot since she cannot borrow from the following period.
$\longrightarrow$loss in welfare.

Fiscal policy becomes powerful:

1. Consider $w_{t+1} \uparrow$ but no consumption smoothing, because she cannot borrow against the future income.
$\longrightarrow$No change in c_t, one-to-one increase in c_{t+1}.

2. The case of $w_t \uparrow$ (plus assumption that borrowing constraint still binds) results one-to-one increase in c_t.



3.3.2 Wealth

Wealth

Assume that the household is born with some wealth h_{t-1} (eg inherited wealth; house). The price of this wealth/asset today (in period t) is q_t, which household takes *as given*. Household may invest more to the same wealth today, ie cumulate wealth *into* h_t. The period t flow budget constraint is

$$c_t + s_t + q_t h_t = w_t + q_t h_{t-1}.$$

Note that it may be written as

$$c_t + s_t + q_t(h_t - h_{t-1}) = w_t,$$

where $h_t - h_{t-1}$ is the change in the stock of the asset h.

Period $t+1$ budget constraint is

$$c_{t+1} + s_{t+1} + q_{t+1}h_{t+1} = w_{t+1} + (1+r_t)s_t + q_{t+1}h_t$$

With terminal condition $h_{t+1} = s_{t+1} = 0$ this reduces to

$$c_{t+1} = w_{t+1} + (1+r_t)s_t + q_{t+1}h_t$$

Solve s_t from $t+1$ budget constraint and collect the terms to obtain the intertemporal budget constraint

$$c_t + \frac{c_{t+1}}{1+r_t} + q_t h_t = w_t + \frac{w_{t+1}}{1+r_t} + q_t h_{t-1} + \frac{q_{t+1}h_t}{1+r_t}$$

Terms:

- $c_t + c_{t+1}/(1+r_t)$ the present discounted value of the stream of consumption
- $q_t h_t$ the present discounted value expenditure on the asset h_t
- $w_t + w_{t+1}/(1+r_t)$ the present discounted value of the income stream
- $q_t h_{t-1}$ existing value of ("inherited") assets. It sells the asset h_{t-1} with price q_t *and buys it back with the same price!*
- $\frac{q_{t+1}h_t}{1+r_t}$ present discounted value of the asset in period $t+1$.

Note: If the household sell all assets h_{t-1} in period t and invests them to the savings instrument s_t, the intertemporal budget constraint is

$$c_t + \frac{c_{t+1}}{1+r_t} = w_t + \frac{w_{t+1}}{1+r_t} + q_t h_{t-1}$$

Then the term $q_t h_{t-1}$ denotes the initial wealth.

3.3.3 Housing investments

Housing investments

This subsection is borrowed from Pertti Haaparanta's course in 2014.

Assume utility time separable

$$U = U(C_t, H_t) + \beta U(C_{t+1}, H_{t+1}),$$

where H_t is the consumption of housing services in period t.

Assume perfect capital markets. The periodic budget constraints are

$$C_t + p_t^I I_t = W_t + D, \quad H_t = \kappa I_t$$
$$C_{t+1} + p_{t+1}^I I_{t+1} = W_{t+1} - (1+r)D$$
$$H_{t+1} = \kappa \left[I_{t+1} + (1-\delta) I_t\right]$$

- I_t is the housing investment (e.g. building a house, buying a flat) in period t,
- $\kappa > 0$ gives the flow of housing services coming from one unit of housing investment,
- p_t^I is the real price of housing.

Combining the budget constraints leads to

$$C_t + p_1^I \frac{H_t}{\kappa} + \frac{1}{1+r}\left[C_{t+1} + p_{t+1}^I \left(\frac{H_{t+1}}{\kappa} - (1-\delta)\frac{H_t}{\kappa}\right)\right] = W_t + \frac{W_{t+1}}{1+r}$$

Notes

- Here housing services H are proportional to the amount of housing I the consumer has bought/built.
- Note the assumption that no rental market for housing services exists.

The first order conditions for the purchase of housing services/investment in housing are

$$U_{H_t} = \lambda \left[\frac{p_t^I}{\kappa} - \frac{p_{t+1}^I (1-\delta)}{\kappa(1+r)}\right]$$
$$\beta U_{H_{t+1}} = \lambda \frac{1}{\kappa(1+r)} p_{t+1}^I$$

Here $U_{H_i} \equiv \frac{\partial U}{\partial H_i}$

- Note the *intertemporal speculation* due to potential changes in price of housing.
- This implies variation over time in both housing investment and consumption of housing services.

But there is also variation in the relative consumption within each period due to changes in housing prices as

$$\frac{U_{C_t}}{U_{H_t}} = \frac{1}{\left[\frac{p_t^I}{\kappa} - \frac{p_{t+1}^I(1-\delta)}{\kappa(1+r)}\right]}$$

3.4 Equilibrium in an Endowment Economy

These slides are based on the textbook by Garín, Lester and Sims (2017).

Model setup

Consider an economy with L households. We index them by j. L is large such that a household j is a price taker. Consider the two-period model with exogenous income $w_t(j)$ and $w_{t+1}(j)$. The savings rate is common to each household. Assume, again, the logaritmic preferences.

Household j problem is to choose $c_t(j)$ and $c_{t+1}(j)$ by maximizing utility

$$\max_{c_t(j), c_{t+1}(j)} U(j) = \log[c_t(j)] + \beta \log[c_{t+1}(j)]$$

subject to flow budget constraints

$$c_t(j) + s(j) = w_t(j)$$
$$c_{t+1}(j) = w_{t+1}(j) + (1+r)s(j).$$

Note that the optimality conditions are identical to each household j:

$$\frac{c_{t+1}(j)}{c_t(j)} = \beta(1+r)$$

since r is the same for each household. This means that *the consumption growth rate* is the same for each household but not the *level of consumption*.

This results the consumption functions

$$c_t(j) = \frac{1}{1+\beta}\left[w_t(j) + \frac{w_{t+1}(j)}{1+r}\right]$$
$$c_{t+1}(j) = \frac{\beta}{1+\beta}(1+r)\left[w_t(j) + \frac{w_{t+1}(j)}{1+r}\right].$$

Competitive equilibrium

Even if the real rate r is given for each household (L is large!), in the aggregate (macro!) level it is *endogenous* and determined as a consequence of equilibrium.

Competitive equilibrium

Competitive equilibrium is a set of prices and quantities for which all agents are behaving optimally and all markets simultaneously clear.

The *price* is r and the market is *financial market* (ie market for loans/bonds). Demand and supply in this market must be equal.

Even if an invidual household may save $s(j) > 0$ or borrow $s(j) < 0$ for given r, *in the aggregate level saving must be zero:*

$$\sum_{j=1}^{L} s(j) = 0.$$

Sum the first period budget constraint over all households, ie aggregate:

$$\sum_{j=1}^{L} c_t(j) + \sum_{j=1}^{L} s(j) = \sum_{j=1}^{L} w_t(j)$$

and impose market clearing condition $\sum_{j=1}^{L} s(j) = 0$ (and notation $c_t = \sum_{j=1}^{L} c_t(j)$ and $c_{t+1} = \sum_{j=1}^{L} c_{t+1}(j)$) obtain *aggregate resource constraint*

$$c_t = w_t.$$

(This is due to the fact that there is no production and, hence, no way to transfer resources between periods t and $t+1$.)

Equilibrium with Identical Households

Assume that

1. preferences are identical, and
2. households face the same income stream $w_t(j)$ and $w_{t+1}(j)$.

To ease calculations, we also normalize the number of households $L = 1$, i.e. households are indexed in $[0, 1]$.

These assumptions give us

$$w_t(j) = w_t \text{ and } w_{t+1}(j) = w_{t+1}, \text{ for all } j.$$

Note

- Consumption functions are the same for each household.

- Market clearing implies $s = 0$
  $\longrightarrow$no household will borrow or lend and the aggregate resource constraint is
  $$c_t = w_t \text{ (and } c_{t+1} = w_{t+1}\text{)}.$$

- We have two unknowns c_t and r and two equations (consumption function and aggregate resource constraint).

In other words, the real interest rate must solve from

$$\frac{w_{t+1}}{w_t} = \beta(1 + r)$$

ie

$$1 + r = \frac{1}{\beta}\frac{w_{t+1}}{w_t}$$

ie real interest rate is a function of (gross) growth rate of economy.

3.5 Infinite horizon

3.5.1 Preliminaries

Intertemporal utility

In the previous section the planning horizon was only two periods: "today" and "tomorrow".

Next we allow for infinite horizon:

- Households are "dynasties" who live forever

- Decision-making takes into account the future generations.

- Households value today's consumption more than future consumption, time preference rate is ρ and the discount rate is $\beta \equiv 1/(1 + \rho)$.

- Instantaneous utility is given by
  $$U_t = U(c_t)$$
  with the usual concavity assumptions: $U_t' > 0$ and $U_t'' \leq 0$.

The consumption decision

The representative household seeks to maximize the present value of utility

$$\max_{\{c_{t+s}, a_{t+1+s}\}_{s=0}^{\infty}} V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) \tag{3.7}$$

subject to the *flow* budget constraints

$$a_{t+1+s} + c_{t+s} = w_{t+s} + (1 + r_{t+s})a_{t+s}, \quad s = 0, 1, 2, \ldots \tag{3.8}$$

where

c_t is consumption in period t

$U(c_t)$ is the instantaneous utility (related assumptions are listed above).

β is the discount factor (defined above), $0 < \beta < 1$.

a_t is the net stock of financial assets *at the beginning of the period t*. Their level has been decided in the previous period!

If $a_t > 0$ households are net lenders, if $a_t < 0$ households are net borrowers.

Hence, since a_{t+1} denotes the assets held at the beginning of the period $t+1$, it corresponds the savings s_t in the two-period model.

r_t is the interest rate on financial assets a_t from period $t-1$ to period t and is paid *at the beginning* of the period.

w_t is household income; it is assumed to be *exogenous* at this stage.

Budget constraint

At the beginning of the period t, the *stock* of financial assets a_t is given.

In period t, households must choose the entire future path of consumption and savings:

$\{c_t, a_{t+1}\}$ of the period t

$\{c_{t+1}, a_{t+2}\}$ of the period $t+1$

$\vdots$

Due to the linking of consumption and financial assets this is equivalent for choosing the entire path of consumption $\{c_t, c_{t+1}, c_{t+2}, \ldots\}$

Note that the budget constraint is now in the *flow* form.

Optimality conditions

The solution can be obtained by the Lagrange multiplier method:

$$\mathcal{L} = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}) + \lambda_{t+s} [w_{t+s} + (1 + r_{t+s})a_{t+s} - c_{t+s} - a_{t+s+1}]\} .$$

Since we have period-by-period (=flow) budget constraint*s*, there is a separate Lagrange multiplier for each period!

The optimality (first order) conditions are as follows:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0 \quad s = 0, 1, 2, \dots \\ \frac{\partial \mathcal{L}}{\partial a_{t+s+1}} &= -\lambda_{t+s} + \lambda_{t+s+1}(1 + r_{t+s+1}) = 0 \quad s = 0, 1, 2, \dots. \end{aligned}$$

Solve λ_{t+s} and λ_{t+1+s} from the first set of FOCs and substitute them to the second set of FOCs to obtain the *Euler equation*:

$$\frac{\beta U'(c_{t+1+s})}{U'(c_{t+s})}(1 + r_{t+1+s}) = 1, \quad s = 0, 1, 2, \dots.$$

This applies to any $t \in \{-\infty, \dots, 0, \dots, \infty\}$ such that we also write it as

$$\frac{\beta U'(c_{t+1})}{U'(c_t)}(1 + r_{t+1}) = 1, \quad t = -\infty, \dots, -1, 0, 1, 2, \dots, \infty.$$

Is is identical to that of the two-period model, but applied to all possible s!

Intertemporal budget constraint

As was written above we may eliminate the assets $\{a_t, a_{t+1}, a_{t+2}, \dots\}$. This is done by iterating the budget constraint. Lets write it for period t and $t + 1$:

$$\begin{aligned} a_{t+1} + c_t &= w_t + (1 + r_t)a_t \\ a_{t+2} + c_{t+1} &= w_{t+1} + (1 + r_{t+1})a_{t+1}. \end{aligned}$$

Solve a_{t+1} from the period t budget constraint and substitute it into period $t + 1$ budget constraint to obtain

$$a_{t+2} + c_{t+1} + (1 + r_{t+1})c_t = w_{t+1} + (1 + r_{t+1})w_t + (1 + r_{t+1})(1 + r_t)a_t.$$

This can be written (divide both sides by $1 + r_{t+1}$) as

$$\frac{a_{t+2}}{1 + r_{t+1}} + \frac{c_{t+1}}{1 + r_{t+1}} + c_t = \frac{w_{t+1}}{1 + r_{t+1}} + w_t + (1 + r_t)a_t.$$

Further substitutions of $a_{t+2}, a_{t+3}, \ldots, a_{t+T}$ will give more general form

$$\frac{a_{t+T}}{\prod_{k=1}^{T-1}(1 + r_{t+k})} + \sum_{s=0}^{T-1} \frac{c_{t+s}}{\prod_{k=1}^{s}(1 + r_{t+k})} = \sum_{s=0}^{T-1} \frac{w_{t+s}}{\prod_{k=1}^{s}(1 + r_{t+k})} + (1 + r_t)a_t, \quad (3.9)$$

where

$$\prod_{k=1}^{s-1}(1 + r_{t+k}) \equiv (1 + r_{t+1})(1 + r_{t+2}) \cdots (1 + r_{t+s-1}).$$

Let $T \to \infty$ to obtain the *infinite intertemporal budget constraint*

$$\sum_{s=0}^{\infty} \frac{c_{t+s}}{\prod_{k=1}^{s}(1 + r_{t+k})} = \sum_{s=0}^{\infty} \frac{w_{t+s}}{\prod_{k=1}^{s}(1 + r_{t+k})} + (1 + r_t)a_t. \quad (3.10)$$

and an additional constraint (corresponding the first term in (3.9)):

$$\lim_{T\to\infty} \beta^T a_{t+T} U'(c_{t+T}) = 0. \quad (3.11)$$

Since by assumptions $U'(c) > 0$, this can be expressed as

$$\lim_{T\to\infty} \beta^T a_{t+T} = 0.$$

This says that in the very distant future discounted value of asset must go to zero. This is also known as *no-Ponzi-game* condition. The economic meaning is that *households cannot finance their consumption indefinitely by borrowing*.

Note that the β^T comes from

$$\beta^T = \frac{1}{\prod_{s=1}^{T-1}(1 + r_{t+s})}.$$

Consumption function

To obtain consumption function, we need to parametrize the utility function. Let's use the constant relative risk aversion (CRRA), ie the CES form as in the two-period case.

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1 + r_{t+1})$$

Shift it one period forward

$$\left(\frac{c_{t+2}}{c_{t+1}}\right)^{\sigma} = \beta(1 + r_{t+2})$$

and solve c_{t+2} and substitute it to the first one to obtain

$$\left(\frac{c_{t+2}}{c_t}\right)^{\sigma} = \beta^2(1 + r_{t+1})(1 + r_{t+2})$$

and more generally s period consumption Euler equation:

$$\left(\frac{c_{t+s}}{c_t}\right)^{\sigma} = \beta^s \prod_{k=1}^{s}(1 + r_{t+k}),$$

which gives

$$c_{t+s} = \left[\beta^s \prod_{k=1}^{s}(1 + r_{t+k})\right]^{1/\sigma} c_t.$$

Substitute this to the intertemporal budget constraint (3.10)

$$\sum_{s=0}^{\infty} \frac{\left[\beta^s \prod_{k=1}^{s}(1 + r_{t+k})\right]^{1/\sigma} c_t}{\prod_{k=1}^{s}(1 + r_{t+k})} = W_t \equiv \sum_{s=0}^{\infty} \frac{w_{t+s}}{\prod_{k=1}^{s}(1 + r_{t+k})} + (1 + r_t)a_t.$$

Assuming $\sigma = 1$ above equation simplifies to

$$\sum_{s=0}^{\infty} \beta^s \frac{\left[\prod_{k=1}^{s}(1 + r_{t+k})\right] c_t}{\prod_{k=1}^{s}(1 + r_{t+k})} = W_t \equiv \sum_{s=0}^{\infty} \frac{w_{t+s}}{\prod_{k=1}^{s}(1 + r_{t+k})} + (1 + r_t)a_t.$$

The consumption function as follows

$$c_t = (1 - \beta)W_t = \frac{\rho}{1 + \rho} W_t$$

Features of the consumption function

Only small fraction of the wealth is consumed: suppose $\rho = 0.03$ (3% per annum), then $\rho/(1+\rho) = 0.029$!

Temporary increase in income has a small effect on consumption!

Permanent increase (*income is higher in all future periods*) has a larger effect!

Expected future income increase increase consumption *already today*!

If households are net borrowers $a_t < 0$, increase in r_t will cause c_t to decline.

Savings is undertaken to offset (expected) future falls in income.

What next?

The consumption theory only determines c_t ($t = 0, 1, 2, \dots$).

To study (to endogenize) the wages, we need to determine *the demand for labour* and *supply of labour*

$\longrightarrow$Firms problem to determine the demand for labour

$\longrightarrow$Augment household's problem to determine the labour supply. how these are coordinated!

In addition to labour, firms use capital as a factor of production

$\longrightarrow$Capital formation, ie investments!

All of these results a *real business cycle* model (RBC).

Finally we will study the nominal economy

$\longrightarrow$monetary policy!

4 Labour markets

4.1 Supply of labour

Consumption and leisure

In the previous section we assumed that households worked a fixed time (scaled/normalized to unity).

Time endowment (=amount of time) is unity and divided between leisure l_t and work n_t:

$$n_t + l_t = 1.$$

We may write the instantaneous utility as

$$U(c_t, l_t) = U(c_t, 1 - n_t)$$

with usual assumptions: $U_c > 0$, $U_l < 0$, $U_{cc} \leq 0$, $U_{ll} \leq 0$, $U_n = -U_l$ and the household budget constraint

$$a_{t+1} + c_t = w_t n_t + (1 + r_t) a_t,$$

where w_t is the real wage per unit of time.

Optimality conditions

Household chooses consumption, savings and leisure: $\{c_{t+s}, a_{t+1+s}, n_{t+s};\ s \geq 0\}$. The Lagrangean is the following

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}, 1 - n_{t+s}) \\ + \lambda_{t+s} \left[w_{t+s} n_{t+s} + (1 + r_{t+s}) a_{t+s} - c_{t+s} - a_{t+s+1}\right] \}.$$

and the optimality conditions are

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s U_{c,t+s} - \lambda_{t+s} = 0 \quad s \geq 0$$

$$\frac{\partial \mathcal{L}_t}{\partial n_{t+s}} = -\beta^s U_{l,t+s} + \lambda_{t+s} w_{t+s} = 0 \quad s \geq 0$$

$$\frac{\partial \mathcal{L}_t}{\partial a_{t+1+s}} = \lambda_{t+1+s}(1 + r_{t+1+s}) - \lambda_{t+s} = 0 \quad s \geq 0$$

plus budget constraint.

Set $s = 0$ and eliminate λ_t to get

$$\frac{U_{l,t}}{U_{c,t}} = w_t. \tag{4.1}$$

This express supply of labour n_t as a function of consumption and wage rate. Consumption is derived exactly as before.

Parametric versions

Consider the following instantaneous utility

$$U(c_t, l_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma} + \log l_t,$$

where $\sigma > 0$, then

$$U_{c_t} = c_t^{-\sigma} \text{ and } U_{l_t} = \frac{1}{l_t}.$$

The parametric version of (4.1) is then the following

$$\frac{1/l_t}{c_t^{-\sigma}} = w_t,$$

and supply of labour ($l_t = 1 - n_t$)

$$n_t = 1 - \frac{c_t^{\sigma}}{w_t}.$$

It is easy to see that the supply of labour *ceteris paribus* increases when real wage rises and decreases when consumption increases.

However, note that consumption increases when real wages rise! Hence, the net effect is unclear.

Study the income effect and the subsitution effect!

Another popular parameterization is to put the labour supply directly to the utility function

$$U(c_t, n_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi},$$

where $\sigma \geq 0$ and $\varphi \geq -1$.

Then

$$U_{n_t} = -n_t^{\varphi} \quad U_{c_t} = c_t^{-\sigma}$$

and the optimality conditions as follows

$$c_t^{\sigma} n_t^{\varphi} = w_t$$

A closer look at employment response

Let's rewrite previous equation as follows

$$n_t^{\varphi} = w_t c_t^{-\sigma}$$

The elasticity of substitution is unity (or $1/\varphi$), and wealth elasticity is $-\sigma$ (or $-\sigma/\varphi$).

- The substitution effect dominates the negative wealth effect if $0 < \sigma < 1$ and
- *vice versa* if $\sigma > 1$.
- They cancel each other when $\sigma = 1$ (logarithmic utility).

Intertemporal substitution in labour supply
Two periods and log-preferences

$$U = \log c_1 + b \log(1 - n_1) + \beta \left[\log c_2 + b \log(1 - n_2)\right]$$

subject to lifetime budget constraint

$$c_1 + \frac{1}{1+r} c_2 = w_1 n_1 + \frac{1}{1+r} w_2 n_2$$

Households choose c_1, c_2, n_1 and n_2. Consider the first order conditions for n_1 and n_2.

$$\frac{b}{1-n_1} = \lambda w_1$$
$$\frac{\beta b}{1-n_2} = \frac{1}{1+r} \lambda w_2.$$

Arrange them as

$$\frac{1-n_1}{1-n_2} = \frac{1}{\beta(1+r)} \frac{w_2}{w_1}.$$

Message:

- Relative labour suppy responds to relative wages.
- If $w_1 \uparrow$ (relative to w_2), then $1 - n_1 \downarrow$, ie $n_1 \uparrow$.
- Due to log-preferences, the elasticity of substitution between leisure in the two periods is 1!
- $r \uparrow$ results n_1: rise in r increases the attractiveness of working today and saving relative to working tomorrow.

$\longrightarrow$ "Make the hay while sun shines."

4.2 Demand for labour

Quick visit in the demand for labour
Consider firm with a production function of the homogenous of degree one

$$y_t = f(k_t, n_t),$$

where

k_t is capital input

n_t is labour input

The good y_t is *numeraire* and its price is normalized to unity $p_t = 1$. (All other prices are expressed with respect to this price, ie in relative prices.)

Profits are given

$$\pi_t = y_t - r_t^K k_t - w_t n_t,$$

where

r_t^K is the rental rate of capital, ie the flow cost of capital, and

w_t is the real wage (as before).

Firm operate under perfect competition and takes all prices (also the factor prices) given. The first order conditions equates the marginal productivity of factor to its relative price

$$\frac{\partial y_t}{\partial k_t} = f_{k_t} = r_t^K, \quad \frac{\partial y_t}{\partial n_t} = f_{n_t} = w_t.$$

The second optimality condition gives the *demand for labour*.

Few remarks:

- The homogenous of degree one is same as constant-returns-to-scale.
- Euler's theorem says that for linear-homogenous function the sum of partial derivatives of times the variable with respect to which the derivative was computed equals the original function. Hence,

  $$y_t = \frac{\partial y_t}{\partial k_t} k_t + \frac{\partial y_t}{\partial n_t} n_t$$

  and when combined with optimality conditions

  $$y_t = r_t^K k_t + w_t n_t.$$

  Total output is identical to total factor income. Profits are zero.
- The first optimality condition with respect to capital gives the demand for capital. We move forward to analyze capital formation, ie investments

Example: Cobb-Douglas

Let's assume that the firms' production function is of a Cobb-Douglas form

$$y_t = k_t^\alpha (A_t n_t)^{1-\alpha} \quad \forall t > 0, \tag{4.2}$$

where

y_t is the output (real),

k_t is the amount of capital

n_t is the amount of labour (hours/persons)

A_t is the aggregate level of technology

w_t will be the real wages per unit of work (hours/persons)

r_t will be the rental rate of capital (again real)

Firms choose amount of capital k_t and labour n_t by maximizing profits

$$\pi_t = y_t - r_t^K k_t - w_t n_t,$$

subject to the production technology (4.2).

The optimality conditions are

$$\frac{\partial \pi_t}{\partial n_t} = (1-\alpha) k_t^\alpha (A_t n_t)^{-\alpha} A_t - w_t = (1-\alpha)\frac{y_t}{n_t} - w_t = 0$$
$$\frac{\partial \pi_t}{\partial k_t} = \alpha k_t^{\alpha-1} (A_t n_t)^{1-\alpha} - r_t^K = \alpha \frac{y_t}{k_t} - r_t^K = 0.$$

Hence

$$(1-\alpha)\frac{y_t}{n_t} = w_t$$
$$\alpha \frac{y_t}{k_t} = r_t^K.$$

5 Dynamic theory of taxation

5.1 Government budget constraint

Government accounts

Table 18B

REVENUE AND EXPENDITURE OF GENERAL GOVERNMENT, EXCESSIVE DEFICIT PROCEDURE (BASED ON ESA 1995)

Finland **(percentage of GDP at current market prices (excessive deficit procedure))**

	1980	1985	1990	1995	2000	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
1. Taxes on production and imports	13.1	14.3	15.0	13.5	13.5	13.8	13.8	13.7	13.0	12.9	13.4	13.4	14.2	14.4	14.8	15.1	15.3
2. Current taxes on income and wealth	14.2	16.5	17.3	17.3	21.1	17.5	17.5	17.3	17.5	17.4	16.1	16.0	16.4	16.0	16.9	16.7	16.8
3. Social contributions	10.7	11.3	12.7	14.7	12.1	11.8	12.1	12.4	12.0	12.2	12.9	12.8	12.7	13.3	13.4	13.6	13.7
4. Of which actual social contributions	9.2	9.6	11.9	14.5	12.1	11.8	12.1	12.4	12.0	12.2	12.9	12.8	12.7	13.3	13.4	13.6	13.7
5. Other current revenue, including sales	5.9	7.6	8.4	9.8	8.3	9.0	9.1	9.5	9.8	10.6	10.5	10.4	10.4	10.4	10.4	10.6	10.8
6. Total current revenue	43.9	49.7	53.4	55.2	55.0	52.1	52.6	52.9	52.4	53.1	53.0	52.6	53.7	54.1	55.6	55.9	56.6
7. Government comsumption expenditure	18.4	20.6	21.8	22.7	20.6	22.2	22.5	22.2	21.5	22.5	25.2	24.7	24.5	25.1	25.7	25.7	25.6
8. Collective consumption	7.1	7.5	7.8	8.5	7.5	7.7	7.8	7.6	7.4	7.7	8.6	8.2	8.0	8.2	8.5	8.6	8.6
9. Social transfers in kind	11.3	13.0	14.0	14.2	13.1	14.5	14.7	14.7	14.2	14.8	16.6	16.5	16.4	16.9	17.2	17.2	17.0
10. Compensation of employees	12.5	14.4	14.8	15.1	13.1	13.6	13.8	13.5	12.9	13.3	14.8	14.5	14.2	14.5	14.7	14.6	14.4
11. Intermediate consumption	6.4	7.2	7.8	8.9	8.1	9.2	9.4	9.4	9.3	10.0	11.4	11.5	11.5	11.8	12.1	12.4	12.6
12. Social transfers other than in kind	10.9	13.8	14.6	21.9	16.2	16.6	16.5	16.0	15.1	15.3	18.2	18.3	18.0	18.8	19.7	20.1	19.9
13. Social transfers in kind via market producers	0.9	0.9	1.1	1.4	1.6	2.1	2.2	2.2	2.1	2.3	2.7	2.7	2.7	2.7	2.9	2.9	2.8
14. Interest	1.0	1.8	1.4	3.9	2.8	1.6	1.5	1.4	1.5	1.4	1.1	1.1	1.1	1.0	1.0	1.0	1.0
15. Subsidies	3.1	3.0	2.8	2.7	1.5	1.3	1.3	1.4	1.3	1.3	1.4	1.5	1.4	1.4	1.4	1.4	1.4
16. Other current expenditure	1.1	1.4	1.7	2.0	2.3	2.5	2.6	2.6	2.5	2.6	3.0	3.0	3.0	3.0	3.2	3.2	3.1
17. Total current expenditure	35.9	42.5	44.2	56.0	45.5	46.9	47.3	46.5	44.7	46.3	52.6	52.6	51.9	53.4	55.0	55.5	55.2
18. Gross saving	8.0	7.1	9.3	-0.8	9.4	5.2	5.3	6.3	7.7	6.8	0.5	0.0	1.8	0.8	0.6	0.5	1.4
19. Capital transfers received	0.1	0.3	0.2	0.2	0.4	0.4	0.4	0.4	0.4	0.5	0.4	0.4	0.3	0.3	0.4	0.4	0.4
20. Total revenue	44.0	49.9	53.6	55.4	55.4	52.5	53.0	53.3	52.7	53.6	53.4	53.0	54.1	54.5	56.0	56.3	57.0
21. Gross fixed capital formation	3.5	3.4	3.5	2.6	2.4	2.8	2.5	2.3	2.4	2.5	2.8	2.5	2.5	2.6	2.8	2.9	2.8
22. Other capital expenditure	0.7	0.6	0.5	2.9	0.4	0.3	0.4	0.2	0.3	0.4	0.5	0.4	0.4	0.3	0.3	0.3	0.3
23. Total expenditure	40.2	46.5	48.2	61.5	48.3	50.0	50.2	49.1	47.4	49.2	55.9	55.5	54.8	56.3	58.1	58.6	58.3
24. Tax burden	36.6	40.5	44.4	46.0	47.4	43.6	44.1	43.9	43.1	43.0	43.0	42.6	43.8	44.2	45.7	45.9	46.3
25. Net lending (+) or net borrowing (-)	3.8	3.4	5.4	-6.1	7.0	2.5	2.9	4.2	5.3	4.4	-2.5	-2.5	-0.7	-1.8	-2.1	-2.3	-1.3

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Link to original table.

In more condensed form

(Most of the material in this section is based on the lecture notes by Dirk Krueger (2005))

Government expenditures:

G_t = Purchases of Goods + Wages and Salaries + Public Investments + other

Net taxes:

T_t = Taxes + Social insurance contributions + Other receipts − Social insurance transfers − Subsidies

Net interests:

$r_{t-1}B_{t-1}$

The government budget constraint

Period 1

$$G_1 = T_1 + B_1$$

Period t

$$G_t + (1+r)B_{t-1} = T_t + B_t$$

$G_t - T_t$ is often called *primary goverment deficit* and

$G_t + rB_{t-1} - T_t$ *goverment deficit*.

Iterate forward:

$$G_2 + (1+r)B_1 = T_2 + B_2$$

or

$$B_1 = \frac{T_2 + B_2 - G_2}{1+r}$$

plug this to period 1 budget constraint to obtain

$$G_1 = T_1 + \frac{T_2 + B_2 - G_2}{1+r}$$

or

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r} + \frac{B_2}{1+r}$$

iterate it T periods

$$G_1 + \frac{G_2}{1+r} + \frac{G_3}{(1+r)^2} + \cdots + \frac{G_T}{(1+r)^{T-1}}$$
$$= T_1 + \frac{T_2}{1+r} + \frac{T_3}{(1+r)^2} + \cdots + \frac{T_T}{(1+r)^{T-1}} + \frac{B_T}{(1+r)^T}.$$

Assume that the government lives forever and — in the limit — the present discounted value of debt is going to be zero

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T B_T = 0,$$

ie government cannot accumulate debt "too fast". The infinite budget constraint will be

$$\sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}}$$

5.2 Ricardian equivalence

Ricardian equivalence

- Some of the most interesting and important discussions in macroeoconomics concerns the status of government debt.
- If I hold government debt, can I really include it in my wealth?
- Eventually I have to pay taxes to the governemnt for it to be able to pay back the debt (with interest) to me.
- The present value of my tax liabilities then equals the value of government debt in my hands now: I cannot be at all wealthier by holding the debt.
- Is this correct?

- And if it is, does it have implications for the possibilities of the government to affect aggregate demand through changes in its expenditure?

Households pay lump-sum taxes T_t in each period t such that the lifetime budget constraint is

$$c_t + \frac{c_{t+1}}{1+r_t} = w_t - T_t + \frac{w_{t+1} - T_{t+1}}{1+r_t}.$$

This means that w_t is now *gross* (pre-tax) income.

Goverment collect taxes T_t and consumes G_t in period t. Public savings are B. Assume no default risk (by the government)

The period t budget constraint by the government

$$B = T_t - G_t$$

and period $t+1$

$$G_{t+1} = (1+r)B + T_{t+1}.$$

We derive the public sector *intertemporal* budget constraint as the household budget constraint

$$G_t + \frac{G_{t+1}}{1+r} = T_t + \frac{T_{t+1}}{1+r}.$$

It tells that

- The present value of the public expenditure must be equal to the present value tax revenues.
- If the period t budget is in deficit $G_t > T_t$, the second period budget must have surplus. (And *vice versa*.

  For example, if taxes are reduced by one unit ($\Delta T_t = -1$) in the first period, then taxes must be $(1+r)\Delta T_t$ units higher in the second period $t+1$.

Reduce the taxes in the first period by ΔT_1 and add them in period $t+1$ by $\Delta T_1(1+r)$:

$$c_t + \frac{c_{t+1}}{1+r} = w_t - (T_t - \Delta T_1) + \frac{w_{t+1} - [T_{t+1} + (1+r)\Delta T_1]}{1+r}.$$

that reduces to

$$c_t + \frac{c_{t+1}}{1+r} = w_t - T_t + \frac{w_{t+1} - T_{t+1}}{1+r}.$$

Taxes changes has not impact!

Substitute the government budget constraint to the household budget constraint to obtain

$$c_t + \frac{c_{t+1}}{1+r_t} = w_t - T_t + \frac{w_{t+1} - T_{t+1}}{1+r_t}$$
$$= w_t + \frac{w_{t+1}}{1+r} - G_t - \frac{G_{t+1}}{1+r}. \quad (5.1)$$

- The structure of funding of government expenditures does, thus, not matter, what counts for the consumer is just the present value of government expenditure flows. Government bonds are not net wealth and do not affect consumer behaviour.
- This is the Modigliani-Miller -theorem as applied to the public sector funding.
- Ricardian Equivalence holds in the basic versions of the Real Business Cycle (RBC) and New-Keynesian models.
- But we should understand the limitations of the analysis:
- For an individual consumer her discounted tax liabilities need not match the value of government debt she holds: Distributional issues.
- What if taxes are distortionary and not lump-sum as assumed above?
- Governments typically can borrow at interest rates lower than private agents (consumers, firms) or at least at cheaper rates than most of private actors. Implications?
- Capital market imperfections may have an effect, e.g. if private sector is credit constrained.

In recent years there has been a claim that if Ricardian Equivalence holds then temporary debt financed increases in government expenditure cannot increase aggregate demand.

Let us see, so assume that in period t the government increases its expenditure by $dG_t > 0$ while keeping the present value of government expenditure flows unchanged, i.e. $dG_{t+1} = -(1+r)dG_t$.

Then (5.1) then implies that the consumer behavior is not affected at all, $dC_1 = dC_2 = 0$.

The change in aggregate demand equals the sum of changes in private consumption and government expenditure:

$$dC_t + dG_t = 0 + dG_t = dG_t > 0$$

and aggregate demand increases by the amount of government expenditure.

Discussion of the crucial assumptions

1. Frictionless financial markets
   $\longrightarrow$interest rates of lending and borrowing are the same
   $\longrightarrow$interest rates that households and government face are the same.

2. *Absense of binding borrowing constraints*

3. *No generational distribution of debt burden*

5.3 Consumption, labour and capital income taxation

Taxes

We study the taxation issue using a simple two-period consumption and labour supply model with the following modifications

1. Households only work in the first period.

2. In the second period they retire but receive a lump-sum transfer (social security benefit) $b \geq 0$.

3. They pay proportional tax τ_{c_1} on their first period consumption and τ_{c_1} on the second period consumption.

4. They pay proportional tax τ_l on their labour income (only in the first period when they work).

5. They pay proportional tax τ_s on the return from their savings (tax at source).

The maximization problem is the following

$$\max_{c_t, c_{t+1}, s, n} \log(c_t) + \theta \log(1-n) + \beta \log(c_{t+1})$$

subject to

$$(1+\tau_{c_1})c_t + s = (1-\tau_l)wn \tag{5.2}$$

$$(1+\tau_{c_2})c_{t+1} = (1+(1-\tau_s)r)s + b, \tag{5.3}$$

where θ measures how much household value leisure.

As before solve s from (5.3)

$$s = \frac{(1+\tau_{c_1})c_{t+1} - b}{(1+r(1-\tau_s))}$$

and substitute it into (5.2) to obtain intertemporal budget constraint

$$(1+\tau_{c_1})c_t + \frac{(1+\tau_{c_2})c_{t+1}}{(1+r(1-\tau_s))} = (1-\tau_l)wn + \frac{b}{(1+r(1-\tau_s))}.$$

Since we want to derive the first-order condition wrt $1-n$, lets write above such that we have $1-n$ everywhere using the definition $n = 1-(1-n)$

$$(1+\tau_{c_1})c_t + \frac{(1+\tau_{c_2})c_{t+1}}{(1+r(1-\tau_s))} = (1-\tau_l)w(1-(1-n)) + \frac{b}{(1+r(1-\tau_s))}$$

or

$$(1+\tau_{c_1})c_t + \frac{(1+\tau_{c_2})c_{t+1}}{(1+r(1-\tau_s))} + (1-\tau_l)w(1-n) = (1-\tau_l)w + \frac{b}{(1+r(1-\tau_s))}.$$

The Lagrangian

$$\begin{aligned}\mathcal{L} = \log(c_t) + \theta\log(1-n) + \beta\log(c_{t+1}) \\ + \lambda\Big[(1-\tau_l)w + \frac{b}{(1+r(1-\tau_s))} \\ - (1+\tau_{c_1})c_t - \frac{(1+\tau_{c_2})c_{t+1}}{(1+r(1-\tau_s))} - (1-\tau_l)w(1-n)\Big]\end{aligned}$$

The first-order conditions are given by

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial c_t} &= \frac{1}{c_t} - \lambda(1+\tau_{c_1}) = 0 \\ \frac{\partial\mathcal{L}}{\partial c_{t+1}} &= \frac{\beta}{c_{t+1}} - \lambda\frac{(1+\tau_{c_2})}{1+r(1-\tau_s)} = 0 \\ \frac{\partial\mathcal{L}}{\partial(1-n)} &= \frac{\theta}{1-n} - \lambda(1-\tau_l)w = 0\end{aligned}$$

or

$$\frac{1}{c_t} = \lambda(1+\tau_{c_1}) \tag{5.4}$$

$$\frac{\beta}{c_{t+1}} = \lambda\frac{(1+\tau_{c_2})}{1+r(1-\tau_s)} \tag{5.5}$$

$$\frac{\theta}{1-n} = \lambda(1-\tau_l)w \tag{5.6}$$

Solve λ from (5.4) and substitute it to (5.5) to obtain

$$\frac{\beta c_t}{c_{t+1}} = \frac{(1+\tau_{c_2})}{(1+\tau_{c_1})} \frac{1}{1+r(1-\tau_s)}$$

and to (5.6) to obtain

$$\frac{\theta c_t}{1-n} = \frac{(1-\tau_l)w}{1+\tau_{c_1}}.$$

Notes:

- Increase in τ_s reduces after-tax interest rate $1+r(1-\tau_s)$ $\longrightarrow$household will consume more on the first period
- Increase in the first period consumption taxes τ_{c_1} makes household to consume less in the first period (relative to consumption in the second period).
- Increase in the second period consumption taxes τ_{c_2} makes household to consume more in the first period (relative to consumption in the second period).
- Increase in labour taxes τ_l reduces after-tax wage rate and reduces consumption relative to leisure, ie $c_1/(1-n)$ falls. This substitution effect reduces both the first period consumption and labour supply.
- An increase in the first period consumption taxes τ_{c_1} reduces consumption relative to leisure. This is the substitution effect saying that an increase in τ_{c_1} reduces both current period consumption and labour supply.
- Assume tax system where $\tau_l = 0$ and $\tau_{c_1} = \tau_{c_2} = \tau_c$, then there exists a labour income tax τ_l and a lump sum tax T such that for $\tau_c = 0$ household finds it optimal to make exactly the same consumption choices as before.

Why do Americans work so much more than Europeans?

		Labor Supply*		Differences	Prediction Factors	
Period	Country	Actual	Predicted	(Predicted Less Actual)	Tax Rate τ	Consumption/ Output (c/y)
1993–96	Germany	19.3	19.5	.2	.59	.74
	France	17.5	19.5	2.0	.59	.74
	Italy	16.5	18.8	2.3	.64	.69
	Canada	22.9	21.3	–1.6	.52	.77
	United Kingdom	22.8	22.8	0	.44	.83
	Japan	27.0	29.0	2.0	.37	.68
	United States	25.9	24.6	–1.3	.40	.81
1970–74	Germany	24.6	24.6	0	.52	.66
	France	24.4	25.4	1.0	.49	.66
	Italy	19.2	28.3	9.1	.41	.66
	Canada	22.2	25.6	3.4	.44	.72
	United Kingdom	25.9	24.0	–1.9	.45	.77
	Japan	29.8	35.8	6.0	.25	.60
	United States	23.5	26.4	2.9	.40	.74

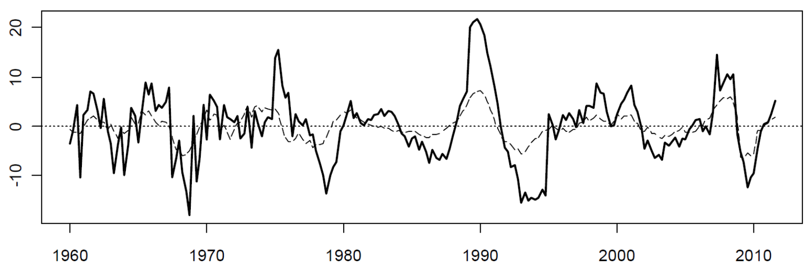
See original article and some other explanations and some Finnish calculations (page 87–94).

6 Investments

6.1 Investments and capital stock

Stylized facts

KUVIO 30: Suomi, yksityiset kiinteät investoinnit



Investments are more volatile than the output. Also *pro-cyclical*

Introduction

(This section is based on the course material by Pertti Haaparanta, 2014.)

- We begin our study of the demand side by focusing on the determinants of the private non-residential capital formation (investment in capital).
- This will not be incorporated into the final model: the model would become too complicated.

- But illustrates one of the transmission channels of monetary policies.
- Traditionally investment has been regarded as one/the major source of economic fluctuations: Keynes and "animal spirits".
- To understand this one must understand the conncetion between investment and stock markets: The q-theory of investment.
- The mainstream theory of private fixed investment is formalization of the following description of incentives to make new fixed investment by Keynes in "General Theory":

  > "The daily revaluations of the Stock Exchange, though they are primarily made to facilitate transfers of old investment between one individual and another, inevitably exert a decisive influence on the rate of current investment. For there is no sense in building up a new enterprise at a cost greater than that at which a similar enterprise can be purchased; whilst there is an inducement to spend on a new project what may seem an extravagant sum, if it can be floated off on the Stock Exchange at an immediate profit."

- James Tobin developed his q-theory of investment on this idea.
- Tobin defined q as the ratio between market value of capital and its replacement costs

$$q = \frac{\text{market value of capital}}{\text{replacement costs}} \tag{6.1}$$

- The idea: if planning to invest in some industry compare the costs of establishing a new firm and making the fixed investment with the costs of buying an established firm.
- Easiest to understand in the context of real estate or housing market.
- The applicability of the idea does not require any stock market, but empirics quite often requires.

Investments

- Theory of investment vs theory of optimal capital stock.

- Assumption: The firm maximizes the present value of its profits (income to firm's owners) = (stock) market value of the firm.
- The profits/period are

$$\Pi_t = AF(K_t) - I_t, \quad F' > 0,\ F'' < 0$$

where K_t = capital stock at the beginning of period t, I_t = investment at the beginning of period t, A = productivity (level of technology).

- Capital stocks and investments are related through the accumulation equation

$$K_{t+1} = K_t + I_t - \delta K_t = I_t + (1-\delta) K_t, \quad 0 < \delta < 1$$

where δ = rate of depreciation (rust and dust)

- Note that net investment is

$$K_{t+1} - K_t = I_t - \delta K_t$$

- The value of the firm to its owners today (period 0) is, with E_0 = expectations formed in period 0

$$V_0 = AF(K_0) - I_0 + \mathrm{E}_0 \frac{V_1}{1+r}$$

Note that we assume interest rate r to be the same in each period.

- By repeated substitution

$$V_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \frac{AF(K_t) - I_t}{(1+r)^t}$$

where r = (real) interest rate, assumed to be constant.

- Assume no uncertainty, *consider the choice of investment in period 0, I_0.*
- First, the choice is equivalent to the choice of K_1 since

$$I_0 = K_1 - (1-\delta) K_0 \tag{6.2}$$

and the same for all future investments, and the value of the firm is

$$V_0 = AF(K_0) - K_1 + (1-\delta) K_0 + \frac{AF(K_1) - K_2 + (1-\delta) K_1}{1+r} + \cdots$$

- Choice of investment in period 0 *is equivalent* to the choice of period 1 capital stock, the choice of investment in period t equivalent to the choice of capital stock in period $t+1$, which gives the optimality conditions

$$\frac{AF_K(K_1) + (1-\delta)}{1+r} = 1$$
$$\frac{AF_K(K_{t+1}) + (1-\delta)}{1+r} = 1$$

  giving

$$AF_K(K_{t+1}) = r + \delta, \quad t = 0, 1, \ldots$$

- $r + \delta$ = the *user cost of capital*, but it is also capital rental rate (remember r_K in previous lecture), the rent (leasing rate) that would arise in case of competitive markets for leasing capital goods.

- In this theory the capital stock is always in its equilibrium level, here the same for all periods, and is given by

$$K^* = F_K^{-1}\left(\frac{r+\delta}{A}\right) \equiv K\left(\frac{r+\delta}{A}\right), \quad K' < 0$$

- This holds also when investment varies between periods e.g. due to productivity changes: The stock is always at the level which would prevail were present conditions to continue to future. E.g. if the total factor productivity A varies over time, the capital stock chosen in period t for period t+1 would be the solution to

$$\frac{A_{t+1}F_K(K_{t+1}) + (1-\delta)}{1+r} = 1$$

  and the change in capital stock would be

$$K\left(\frac{r+\delta}{A_{t+1}}\right) - K\left(\frac{r+\delta}{A_t}\right)$$

- In current period there would not be any need to take into account what happens in the future as there are no costs in changing the capital stock: there is no need to smooth the change over several periods.

- Empirically this is untenable, investment is not this volatile.

6.2 Tobin q

q-theory

- *Assume a stationary environment with identical current and future optimal capital stock*. Then the only investment made is the investment to cover depreciation

$$I = \delta K^*$$

and the value of the firm is

$$V = \sum_{t=0}^{\infty} \frac{AF(K^*) - \delta K^*}{(1+r)^t} = \frac{AF(K^*) - \delta K^*}{1 - \frac{1}{1+r}} =$$

$$\frac{(1+r)}{r}\left(AF(K^*) - \delta K^*\right)$$

- But

$$AF(K^*) - \delta K^* = AF(K^*) - AF_K(K^*)K^* + AF_K(K^*)K^* - \delta K^* =$$

$$\left[\frac{AF(K^*)}{K^*} - AF_K(K^*)\right]K^* + \left[\left((r+\delta) - \delta\right)\right]K^* =$$

$$\left[\left(\frac{AF(K^*)}{K^*} - AF_K(K^*)\right) + r\right]K^*$$

- In the long run equilibrium one expects that *the average product of capital and the marginal product of capital are equal*:

$$\frac{AF(K^*)}{K^*} - AF_K(K^*) = 0$$

by competition and free entry in the markets (easy to see that this holds in the case of the constant returns to scale and the competitive factor markets).

- The value of the firm is then

$$V = \frac{1+r}{r} rK^* \simeq \frac{1}{r} rK^* = K^*$$

- But at the same time the value of the firm must equal the market price of its capital stock

$$V = qK^*$$

which can hold only if $q = \frac{V}{K^*} = 1$. In this case the market value of the capital is equal to itse replacement value (since the price of new capital goods = 1), and as seen, there is no new investment.

- Note also that at the optimum

$$\frac{AF_K(K^*) + (1-\delta)}{1+r} = 1$$

- This is not accidental as the LHS gives the impact of current investment on the value of the firm

$$\frac{AF_K(K^*) + (1-\delta)}{1+r} = \frac{dV}{dK}$$

and under current assumptions

$$q = \frac{V}{K} = \frac{dV}{dK}$$

- But as such, this is a bad theory implying too volatile an investment and incredible fast adjustment of capital stock.

6.3 Adjustment costs

Investments with adjustment costs

The volatility problem known for long, the ways to handle it:

- Jorgenson: Ad hoc, just assume that investment is proportional to the difference between optimal (as implied by the previous theory) and current capital stock.
- Haavelmo: Assume there is a sector producing capital goods with limited capacity similar to that one facing other sectors.
- Lucas: Internal costs of adjustment of capital stock, rationalizes Jorgenson's approach.
- Tobin: Keynes + Lucas (+ Jorgenson), Hayashi (1982) formalization.
- Kydland-Prescott (1982): Time to build.

Assume that the cost of an investment to a firm is *convex*

$$I_t + \frac{\phi}{2}\left(\frac{K_{t+1} - K_t}{K_t}\right)^2 K_t = I_t + \frac{\phi}{2}\left(\frac{I_t - \delta K_t}{K_t}\right)^2 K_t$$

where the assumption, for simplicity, is that the costs of adjustment refer to net changes in capital stock.

The profit of the firm in period t is

$$\Pi_t = AK_t - \left[I_t + \frac{\phi}{2} \left(\frac{I_t - \delta K_t}{K_t} \right)^2 K_t \right] \tag{6.3}$$

and in period $t+1$:

$$\Pi_{t+1} = AK_{t+1} - \left[I_{t+1} + \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 K_{t+1} \right]$$

Note: Now I have assumed right away that production is constant returns to scale.

The firm can be thought of maximizing, in period t, its value by choosing period t investment and period $t+1$ capital stock subject to the constraint

$$I_t + (1-\delta) K_t - K_{t+1} \geq 0$$

The part of the value function relevant for this is

$$\Pi_t + \frac{\Pi_{t+1}}{1+r}$$

and let q_t be the Lagrange-multiplier for the constraint. Then, the Lagrangian is

$$\mathcal{L} = \Pi_t + \frac{\Pi_{t+1}}{1+r} + q_t \left[I_t + (1-\delta) K_t - K_{t+1} \right].$$

The first order conditions for value maximization are

$$I_t : \quad \phi \frac{I_t - \delta K_t}{K_t} = q_t - 1 \tag{6.4}$$

$$K_{t+1} : \quad \frac{1}{1+r} \left[A + \phi\delta \frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} + \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 \right] = q_t$$

- The first equation in (6.4) gives what we want

$$I_t - \delta K_t = \frac{q_t - 1}{\phi} K_t \tag{6.5}$$

  There is net addition to capital stock only if $q > 1$. Also, gross investment in period t is proportional to period t capital stock.

- The second equation in (6.4) implies that the value of current investment depends also on the investments in all other future periods, making investments autocorrelated.
- This equation also tells that q is the impact of an increase in the capital stock on the value of the firm
$$q = \frac{dV}{dK}$$
- But look at (6.3): Per period profit is linearly homogenous in that periods's investment and capital stock (doubling of both doubles the profit).
- (6.5) tells that investment in t is proportional to the capital stock in t.
- But period t capital stock is then proportional to period t-1 capital stock, . . .
- The value of the firm in period 0 must then be proportional to the capital stock of period 0. Repeat this for all periods.
- Thus, we must have
$$V_t = q_t K_t$$
- This is the basis for the empirical work on investments as there are data on market valuations of the firms and replacement costs of capital stock.

6.4 Investment funding

Funding the Investment 1

- Up until now we have assumed that investment is funded by retained earnings.
- Does the structure of funding matter?
- Consider debt. D_t = stock of debt the firm has issued in previous periods, D_{t+1} = the stock of debt it has in the next period after debt issue in period t.

- The period t and $t+1$ profits of the firm are then

$$\Pi_t = AK_t - \left[I_t + \frac{\phi}{2} \left(\frac{I_t - \delta K_t}{K_t} \right)^2 K_t \right] + D_{t+1} - (1+r) D_t$$
$$\Pi_{t+1} = AK_{t+1} - \left[I_{t+1} + \frac{\phi}{2} \left(\frac{I_{t+1} - \delta K_{t+1}}{K_{t+1}} \right)^2 K_{t+1} \right] + D_{t+2} - (1+r) D_{t+1}$$

- The period t debt issuance is chosen to maximize

$$\Pi_t + \frac{\Pi_{t+1}}{1+r}$$

with the impact of debt issuance on the value of the firm being

$$1 - \frac{1+r}{1+r} = 0$$

- Thus, the firm is indifferent between all choices of debt issuance, between no issuance and between issuing so much as to fund the period t investment.
- Thus, debt does not have any impact of the value of the firm.
- The same applies to equity funding: Issuing new equity dilutes the incomes of existing owners, indifference prevails.
- The previous result is known as the Modigliani-Miller -theorem.
- Practice: Lessons from the current crisis and the Finnish crisis in the early 1990's?
- Also theory taking into account the various incentive affects associated with different types of funding.

Investment: Summing Up

- The q-theory tells that investments are affected by the market valuation of the firms relative to the replacement cost of capital (e.g. price of investment goods).

- The market valuation is based on future market prospects of the firm (firm profitability depends on the prices it can sell its goods, future costs of its inputs, etc., many of which depend on the aggregate state of the economy, including its growth).
- Thus, investments are based on expectations which again are affected by new information etc.
- Can create, among others, an accelerator-type of an effect.
- Also the riskiness of the firm's investment matters, as it has an effect on the rate at which markets discount its future impacts and value the firm.
- This rate is also affected by the monetary policy.

7 The real business cycle model

7.1 Introduction

Introduction

We put the pieces of the previous sections together:

- Economy is populated by (infinitely) many households who live forever with preferences over consumption and leisure
- Firms use labour and capital as inputs to produce *one* final good that is either consumed or invested.
- Capital accumulates as in the previous section. No adjustment costs.

We analyze three cases

1. The social planner's outcome
2. Decentralized, competitive economy.
3. *Stochastic* economy and the effect of uncertainty.

We solve the model using numerical techniques.

7.2 Economic environment

Preferences

- Households (infinitely many) has the following preferences

$$U_0 = \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + \theta \log(L_t)\right], \quad 0 < \beta < 1, \tag{7.1}$$

  where

  C_t is consumption in period t

  L_t is leisure in period t

- Each household has time-endowment of one unit that can be allocated to leisure or to work N_t.
- Standard assumptions regarding the periodic utility applies.

Production possibilities

- One good Y_t is produced according to the constant-returns-to-scale Cobb-Douglas technology

$$Y_t = K_t^{\alpha} N_t^{1-\alpha} \quad 0 < \alpha < 1, \tag{7.2}$$

  where

  K_t is the beginning of period t capital stock

  N_t is labour input in period t

Capital accumulation

- The final good can be consumed or invested I_t.
- Investments accumulate capital stock according to

$$K_{t+1} = (1-\delta)K_t + I_t \quad 0 < \delta < 1.$$

- δ is the depreciation rate
- I_t is gross investment
- $I_t - \delta K_t$ is net investments.

Resource constraints

- The size of population is constant over time.
- In each period the choices of consumption, investments, leisure, labour are subject to the resource constraints
- The time allocated to work and leisure cannot exceed the time endowment of one unit:

$$L_t + N_t \leq 1$$

- The total use of the final good, consumption and investments, cannot exceed the output of the final good:

$$C_t + I_t \leq Y_t$$

- Non-negativity constraints: $K_{t+1}, C_t \geq 0$, $0 < L_t, N_t < 1$

7.3 The social planner's outcome

Introduction

Assume that the consumption and production decisions are made by the same agent
$\longrightarrow$benevolent *social planner*

- Social planner maximizes the weighted average of household's utility
- All households have *identical* preferences
$\longrightarrow$objective is to choose sequence $\{C_t, L_t, N_t, Y_t, I_t, K_{t+1}\}_{t=0}^{\infty}$ that maximizes utility (7.1) subject to technological, resource and non-negativity constraints
- taking initial (period 0) capital stock K_0 as given.

Planner's problem

$$\max_{\{C_t,L_t,N_t,Y_t,I_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\log(C_t) + \theta \log(L_t)\right)$$

$$\begin{aligned}
&\text{subject to, for } t = 0, \dots, \infty \\
&L_t + N_t = 1 \\
&C_t + I_t = Y_t \\
&K_{t+1} = (1-\delta)K_t + I_t \\
&Y_t = K_t^{\alpha} N_t^{1-\alpha} \\
&K_{t+1}, C_t \geq 0 \\
&0 < L_t, N_t < 1 \\
&\text{given } K_0.
\end{aligned}$$

The constraints hold equally since there is no *wasting*.

Note that the constraints have to hold in every period $t = 0, \dots, \infty$ such that there is a *sequence* of constraints.

The problem can be simplified by

- solving $L_t = 1 - N_t$ from the first constraint
- and

$$C_t = Y_t - I_t = \underbrace{K_t^{\alpha} N_t^{1-\alpha}}_{=Y_t} - \underbrace{K_{t+1} + (1-\delta)K_t}_{=I_t}. \tag{7.3}$$

The problem reduces to

$$\max_{\{N_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\log\left(K_t^{\alpha} N_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t\right) + \theta \log(1 - N_t)\right]$$

$$\begin{aligned}
&\text{subject to} \\
&K_{t+1}, C_t \geq 0 \\
&0 < L_t, N_t < 1 \\
&\text{given } K_0.
\end{aligned}$$

Keep in mind that the solution to the social planner's problem is a *sequence* $\{N_t, K_{t+1}\}_{t=0}^{\infty}$ that is determined *in period* 0!

The first order conditions (for an interior solution) are

$$K_{t+1}: \quad -\frac{1}{C_t} + \beta\left[\frac{1}{C_{t+1}}\left(\alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right)\right] = 0 \tag{7.4}$$

$$N_t: \quad -\theta\frac{1}{1-N_t} + \frac{1}{C_t}(1-\alpha)\frac{Y_t}{N_t} = 0. \tag{7.5}$$

For $t = 0, \ldots, \infty$. Note that we have utilized the equation (7.3). These equations describe a second order nonlinear difference equation in K for which there are infinitely many solutions for given K_0. We need to an additional boundary condition to pin down the unique solution (optimum). This is the following transversality condition

$$\lim_{T \to \infty} \beta^T \frac{1}{C_T} K_{T+1} = 0.$$

It says that the discounted utility value of the limiting capital stock is zero.

7.4 Competitive equilibrium

Introduction

The allocation (7.4) and (7.5) chosen by social planner can be interpreted as a prediction what would happen in the competitive market economy.

If so, the decentralized economy would be Pareto optimal.

This equivalence requires that conditions underlying the fundamental welfare theorem are satisfied: no taxes, perfect competition, absense of other distortions.

In the decentralized economy,

- each period households sell labour and capital services to firms, and
- buy consumption goods produced by firms, consuming some and accumulating rest as capital and shares.
- Trades take place in competitive markets, which must clear in every period.
- The period t price of the final consumption good is p_t,
- The price of one hour of labour services in terms of the final consumption good in period t is w_t (real wage).,
- The price of renting one unit of capital in terms of the final consumption good is r_t (rental rate of capital).

Firms

Assume that the economy is populated by J firms (J is a fixed number). The value of a particular firm j is

$$Q_t^j = (q_t^j + \pi_t^j) S_t^j, \tag{7.6}$$

where

π_t^j is the divident or profit per share and

q_t^j is the share price.

S_t^j is total number of shares issued by firm j. We can have $S_t^j = 1$ without loss of generality.

The firm's profit in period t is given by

$$\pi_t^j = Y_t^j - r_t K_t^j - w_t N_t^j, \tag{7.7}$$

and production is subject to technological constraint (7.2)

$$Y_t^j = (K_t^j)^\alpha (N_t^j)^{1-\alpha}.$$

The *firm's problem* is to choose Y_t^j, K_t^j, N_t^j in every period t that maximizes value of firm (7.6) subject to technological constraints and appropriate non-negativity constraints, taking prices p_t, wages w_t, and real interest rate r_t and share prices q_t^j given. Due to above assumption and since there is no intertemporal dimension in the decision, it is equivalent in maximizing profits (7.7).

Therefore, the firms' problem is the following

$$\begin{aligned} &\max_{\{N_t^j, K_t^j\}_{t=0}^\infty} \quad \pi_t^j = Y_t^j - r_t K_t^j - w_t N_t^j \\ &\text{subject to} \\ &Y_t^j = (K_t^j)^\alpha (N_t^j)^{1-\alpha}. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} &N_t : \quad (1-\alpha)(N_t^j)^{-\alpha}(K_t^j)^\alpha = w_t \\ &K_t : \quad \alpha (K_t^j)^{\alpha-1}(N_t^j)^{1-\alpha} = r_t \\ &\text{for all } t = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, J. \end{aligned}$$

Households

Total number of households is I.

Each household i has identical initial endowment K_0, and shares s_0^j in every firm j, and identical preferences

$$U^i = \sum_{t=0}^{\infty} \beta^t \left(\log(C_t^i) + \theta \log(1 - N_t^i) \right). \tag{7.8}$$

In each period $t = 0, 1, \ldots, \infty$, the household faces the budget constraint

$$C_t^i + K_{t+1}^i - (1-\delta) K_t^i + \sum_{j=1}^{J} q_t^j s_{t+1}^{ij} = w_t N_t^i + r_t K_t^i + \sum_{j=1}^{J} (q_t^j + \pi_t^j) s_t^{ij} \tag{7.9}$$

The *household's i* problem is to choose the sequence of

$$\{C_t^i, K_{t+1}^i, N_t^i, s_{t+1}^{i1}, \ldots, s_{t+1}^{iJ}\}_{t=0}^{\infty}$$

subject to the budget constraint and non-negativity constraints, taking as given the prices

$$\{p_t, r_t, w_t, q_t^1, \ldots, q_t^J\}_{t=0}^{\infty}$$

and divident streams

$$\{\pi_t^1, \ldots, \pi_t^J\}_{t=0}^{\infty}$$

and *initial endowment* $K_0, s_0^1, \ldots, s_0^J$.

Therefore

$$\max_{\{C_t^i, K_{t+1}^i, N_t^i, s_{t+1}^{i1}, \ldots, s_{t+1}^{iJ}\}_{t=0}^{\infty}} U^i = \sum_{t=0}^{\infty} \beta^t \left(\log(C_t^i) + \theta \log(1 - N_t^i) \right)$$

subject to

$$C_t^i + K_{t+1}^i - (1-\delta) K_t^i + \sum_{j=1}^{J} q_t^j s_{t+1}^{ij} = w_t N_t^i + r_t K_t^i + \sum_{j=1}^{J} (q_t^j + \pi_t^j) s_t^{ij}.$$

The Lagrangian

$$\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \Bigg\{ & \beta^t \left(\log(C_t^i) + \theta \log(1 - N_t^i) \right) \\
& + \lambda_t \Big[w_t N_t^i + r_t K_t^i + \sum_{j=1}^{J} (q_t^j + \pi_t^j) s_t^{ij} - C_t^i \\
& - K_{t+1}^i + (1-\delta) K_t^i - \sum_{j=1}^{J} q_t^j s_{t+1}^{ij} \Big] \Bigg\}
\end{aligned}$$

and the first order conditions (I drop the superscript i for notational simplicity):

$$\begin{aligned}
&C_t: \quad \beta^t \frac{1}{C_t} - \lambda_t = 0 \\
&K_{t+1}: \quad -\lambda_t + \lambda_{t+1}\left(r_{t+1} + (1-\delta)\right) = 0 \\
&N_t: \quad \beta^t(-1)\theta \frac{1}{1-N_t} + \lambda_t w_t = 0 \\
&s^1_{t+1}: \quad \lambda_t(-q^1_t) + \lambda_{t+1}(q^1_{t+1} + \pi^1_{t+1}) = 0 \\
&\vdots \\
&s^J_{t+1}: \quad \lambda_t(-q^J_t) + \lambda_{t+1}(q^J_{t+1} + \pi^J_{t+1}) = 0 \\
&\lambda_t: \quad \text{the budget constraint}
\end{aligned}$$

These are applied for all $t = 0, 1, 2, \ldots.$ or

$$\begin{aligned}
&\beta^t \frac{1}{C_t} = \lambda_t \\
&\lambda_t = \lambda_{t+1}\left(r_{t+1} + 1 - \delta\right) \\
&\beta^t \theta \frac{1}{1-N_t} = \lambda_t w_t \\
&\lambda_t q^1_t = \lambda_{t+1}(q^1_{t+1} + \pi^1_{t+1}) \\
&\vdots \\
&\lambda_t q^J_t = \lambda_{t+1}(q^J_{t+1} + \pi^J_{t+1}) \\
&\lambda_t: \quad \text{the budget constraint.}
\end{aligned}$$

Substituting λ_t and λ_{t+1} (and reintroducing the superscript i) gives

$$\begin{aligned}
&1 = \beta \frac{C^i_t}{C^i_{t+1}}\left(r_{t+1} + 1 - \delta\right) \\
&\theta \frac{1}{1-N^i_t} = \frac{1}{C^i_t} w_t \\
&q^j_t = \beta \frac{C^i_t}{C^i_{t+1}}\left(q^j_{t+1} + \pi^j_{t+1}\right) \quad j = 1, \ldots, J
\end{aligned}$$

Discussion

- The first one is the pricing equation for capital
- The second one defines labour supply
- The last one is the pricing equation for a share j.

Competitive equilibrium

A *competitive equilibrium* is described by the allocations

$$\{C_t^i, K_{t+1}^i, N_t^i, s_{t+1}^{i1}, \ldots, s_{t+1}^{iJ}\}_{t=0}^{\infty} \qquad i = 1, \ldots, I$$

and

$$\{Y_t^j, K_t^j, N_t^j\}_{t=0}^{\infty} \qquad j = 1, \ldots, J$$

and prices

$$\{p_t, r_t, w_t, q_t^1, \ldots, q_t^J\}_{t=0}^{\infty}$$

such that

- the allocations solve the household's problem for all i
- the allocations solve the firms' problem for all j
- in every period t all markets clear, i.e.

$$\sum_{j=1}^{J} Y_t^j = \sum_{i=1}^{I} \left(C_t^i + K_{t+1}^i - (1-\delta) K_t^i \right)$$

$$\sum_{i=1}^{I} N_t^i = \sum_{j=1}^{J} N_t^j$$

$$\sum_{i=1}^{I} K_t^i = \sum_{j=1}^{J} K_t^j$$

$$\sum_{i=1}^{I} s_t^{ij} = 1, \quad j = 1, \ldots, J.$$

The first one gives the *aggregate resource constraint*:

$$Y_t = C_t + I_t,$$

where the symbols without superscript denote aggregate (economy wide) variables.

Given the definition of competitive equilibrium, the following set of equations characterize the system (but does provide the solution as such).

$$
\begin{aligned}
&1 = \beta \frac{C_t}{C_{t+1}} \left(r_{t+1} + 1 - \delta\right) \\
&\theta \frac{1}{1 - N_t} = \frac{1}{C_t} w_t \\
&q_t^j = \beta \frac{C_t}{C_{t+1}} \left(q_{t+1}^j + \pi_{t+1}^j\right) \quad j = 1, \ldots, J \\
&(1-\alpha)(N_t)^{-\alpha}(K_t)^{\alpha} = w_t \\
&\alpha(K_t)^{\alpha-1}(N_t)^{1-\alpha} = r_t \\
&Y_t = C_t + I_t \\
&K_{t+1} = (1-\delta)K_t + I_t.
\end{aligned}
$$

7.5 Uncertainty

Introduction

The above model *without uncertainty* does not really explain fluctuations.

We add uncertainty to have the *real business cycle* model.

The uncertainty is related to the level of technology (this is the reason for the word *real*, ie non-monetary)!

Shocking examples: tax rates, gov. spending, tastes, regulation, terms-of-trade, energy prices,...

Technology shock

Production function

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}.$$

We allow temporary changes in the total factor productivity A_t. Denote $a_t \equiv \log(A_t)$. The typical specification is

$$a_t = \rho a_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid}(0, \sigma^2), \quad |\rho| < 1,$$

where "iid" means identically, indepently distributed. This means, among other things, that $\mathrm{E}_t\, \varepsilon_{t+i} = 0$ ($i > 0$). We call a_t *technology shock* and ε_t *shock innovation*. Note that the unconditional mean of a_t is zero, ie $a = 0$ and that of the level is one, $A = e^0 = 1$.

Planner's problem

$$\max_{\{N_t,K_{t+1}\}_{t=0}^{\infty}} \mathrm{E}\Big\{\sum_{t=0}^{\infty}\beta^t\big[\log\Big(A_tK_t^{\alpha}N_t^{1-\alpha}-K_{t+1}+(1-\delta)K_t\Big)$$

$$+\theta\log(1-N_t)\big]|\mathcal{F}_0\Big\}$$

subject to

$K_{t+1}, C_t \geq 0$

$0 < L_t, N_t < 1$

given A_0, K_0.

$\mathcal{F}_0$ denotes the information set at the time 0, more generally information set at time t is $\mathcal{F}_t = \{A_t, A_{t-1}, \ldots, A_0\}$. Hence, $\mathrm{E}(\cdot|\mathcal{F}_0)$ is the *conditional expectation* wrt information known at time 0. Short-cut notation for this is $\mathrm{E}_0(\cdot)$.

The first order conditions (for an interior solution) are

$$K_{t+1}: \quad -\frac{1}{C_t}+\beta\,\mathrm{E}_t\left[\frac{1}{C_{t+1}}\left(A_{t+1}\alpha\left(\frac{N_{t+1}}{K_{t+1}}\right)^{1-\alpha}+1-\delta\right)\right]=0 \quad (7.10)$$

$$N_t: \quad -\theta\frac{1}{1-N_t}+\frac{1}{C_t}(1-\alpha)A_t\left(\frac{K_t}{N_t}\right)^{\alpha}=0, \quad (7.11)$$

where

$$C_t = A_tK_t^{\alpha}N_t^{1-\alpha}-K_{t+1}+(1-\delta)K_t.$$

Solving the model

This is a nonlinear stochastic difference equation. In general, they are difficult to solve

$\longrightarrow$Our model cannot be solved by "paper-and-pencil"

We will work with an approximate solution that is simpler to solve:

1. Set up the deterministic system of equation
2. Find the deterministic steady-state
3. Calculate the first-order Taylor approximation at the deterministic steady-state.
4. Compute the solution (using one of the many alternative ways)

The system is characterized by the following set of equations:

$$
\begin{aligned}
&\frac{1}{C_t} = \beta\,\mathrm{E}_t\left[\frac{1}{C_{t+1}}\left(\alpha e^{a_{t+1}}\left(\frac{N_{t+1}}{K_{t+1}}\right)^{1-\alpha}+1-\delta\right)\right]\\
&\theta\frac{1}{1-N_t} = \frac{1}{C_t}(1-\alpha)e^{a_t}\left(\frac{K_t}{N_t}\right)^{\alpha}\\
&C_t = e^{a_t}K_t^{\alpha}N_t^{1-\alpha}-K_{t+1}+(1-\delta)K_t\\
&a_t = \rho a_{t-1}+\varepsilon_t.
\end{aligned}
$$

Step 1. deterministic version

"Drop" the expectation operator and the shock process:

$$
\begin{aligned}
&\frac{1}{C_t} = \beta\left[\frac{1}{C_{t+1}}\left(\alpha\left(\frac{N_{t+1}}{K_{t+1}}\right)^{1-\alpha}+1-\delta\right)\right]\\
&\theta\frac{1}{1-N_t} = \frac{1}{C_t}(1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha}\\
&C_t = K_t^{\alpha}N_t^{1-\alpha}-K_{t+1}+(1-\delta)K_t.
\end{aligned}
$$

Step 2. deterministic steady-state

Steady-state is the point where the system converges to in the absense of shocks.

Denote the steady-state point of an arbitrary variable x_t as x (without time-sub-script).

$$
\begin{aligned}
&\frac{1}{C} = \beta\left[\frac{1}{C}\left(\alpha\left(\frac{N}{K}\right)^{1-\alpha}+1-\delta\right)\right]\\
&\theta\frac{1}{1-N} = \frac{1}{C}(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}\\
&C = K^{\alpha}N^{1-\alpha}-\delta K.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
K &= \frac{\mu}{\Omega + \varphi\mu} \\
N &= \varphi K \\
C &= \Omega K \\
Y &= K^{\alpha} N^{1-\alpha},
\end{aligned}
$$

where

$$
\varphi = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right], \qquad \Omega = \varphi^{1-\alpha} - \delta, \qquad \mu = \frac{1}{\varphi}(1-\alpha)\varphi^{-\alpha}.
$$

Step 3. the first-order Taylor approximation

Steps here

1. Take natural logarithms of the system of nonlinear difference equations
2. Linearize at a particular point (typically at the deterministic steady-state)
3. Simplify to get have a system of linear difference equations where the variables of interest are percentage deviations at a point (steady-state)

Linearization is nice since we have simple and efficient tools to solve the system of linear difference equations.

As an example, consider some arbitrary univariate function $f(x_t)$. Taylor expansion around some arbitrary point $x^\star$ (where $x^\star$ belong to possible values of x_t):

$$
f(x_t) \approx f(x^\star) + f'(x^\star)(x_t - x^\star),
$$

where

$$
f'(x^\star) = \frac{\partial f(x_t)}{\partial x_t}\Big|_{x_t = x^\star}.
$$

This generalizes to multivariate functions $f(x_t, y_t)$. The first-order Taylor approximation around the point $(x^\star, y^\star)$

$$
f(x_t, y_t) \approx f(x^\star, y^\star) + f_x(x^\star, y^\star)(x_t - x^\star) + f_y(x^\star, y^\star)(y_t - y^\star),
$$

where f_x denotes the partial derivative of the function with respect to x_t (and similarly to y_t).

Suppose the following non-linear function

$$f(x_t) = \frac{g(x_t)}{h(x_t)}.$$

To log-linearize it, first take natural logarithms of both sides:

$$\log f(x_t) = \log g(x_t) - \log h(x_t)$$

and use the first-order Taylor expansion:

$$\begin{aligned}
\log f(x_t) &\approx \log f(x^\star) + \frac{f'(x^\star)}{f(x^\star)}(x_t - x^\star) \\
\log g(x_t) &\approx \log g(x^\star) + \frac{g'(x^\star)}{g(x^\star)}(x_t - x^\star) \\
\log h(x_t) &\approx \log h(x^\star) + \frac{h'(x^\star)}{h(x^\star)}(x_t - x^\star).
\end{aligned}$$

The above follows from

$$\frac{\partial \log f(x)}{\partial x} = \frac{f'(x)}{f(x)}.$$

Collect everything together

$$\begin{aligned}
&\log f(x^\star) + \frac{f'(x^\star)}{f(x^\star)}(x_t - x^\star) \\
&\qquad = \log g(x^\star) + \frac{g'(x^\star)}{g(x^\star)}(x_t - x^\star) - \log h(x^\star) - \frac{h'(x^\star)}{h(x^\star)}(x_t - x^\star)
\end{aligned}$$

or

$$\begin{aligned}
&\log f(x^\star) + \frac{f'(x^\star)}{f(x^\star)}(x_t - x^\star) \\
&\qquad = \log g(x^\star) - \log h(x^\star) + \frac{g'(x^\star)}{g(x^\star)}(x_t - x^\star) - \frac{h'(x^\star)}{h(x^\star)}(x_t - x^\star).
\end{aligned}$$

Since

$$\log f(x^\star) = \log g(x^\star) - \log h(x^\star)$$

the constant terms cancel out

$$\frac{f'(x^\star)}{f(x^\star)}(x_t - x^\star) = \frac{g'(x^\star)}{g(x^\star)}(x_t - x^\star) - \frac{h'(x^\star)}{h(x^\star)}(x_t - x^\star).$$

To express it as percentage deviation, multiply and divide everything by $x^\star$.

$$\frac{x^\star f'(x^\star)}{f(x^\star)}\frac{(x_t-x^\star)}{x^\star}=\frac{x^\star g'(x^\star)}{g(x^\star)}\frac{(x_t-x^\star)}{x^\star}-\frac{x^\star h'(x^\star)}{h(x^\star)}\frac{(x_t-x^\star)}{x^\star}.$$

Denote $\hat{x}_t \equiv \frac{x_t-x^\star}{x^\star}$ to write above in a more compact form

$$\frac{x^\star f'(x^\star)}{f(x^\star)}\hat{x}_t=\frac{x^\star g'(x^\star)}{g(x^\star)}\hat{x}_t-\frac{x^\star h'(x^\star)}{h(x^\star)}\hat{x}_t.$$

Example: Cobb-Douglas production function

$$Y_t = A_t K_t^\alpha N_t^{1-\alpha}$$

Steady state version

$$Y = \underbrace{A}_{=e^0=1} K^\alpha N^{1-\alpha}.$$

since

$$a = \rho a + 0 \iff (1-\rho)a = 0 \iff a = 0.$$

and $A = e^a$.

Take logs of the original

$$\log Y_t = \log A_t + \alpha \log K_t + (1-\alpha)\log N_t$$

and do the first-order Taylor series expansion

$$\log Y + \frac{1}{Y}(Y_t - Y) = a_t + \alpha \log K + \alpha \frac{1}{K}(K_t - K) + (1-\alpha)\log N + (1-\alpha)\frac{1}{N}(N_t - N). \quad (7.12)$$

Due to the steady-state, the constant terms cancel and we may use the "hat" notation to write

$$\hat{Y}_t = a_t + \alpha \hat{K}_t + (1-\alpha)\hat{N}_t.$$

Log-linearizing the first equation

$$\frac{1}{C_t} = \beta\,\mathrm{E}_t\left[\frac{1}{C_{t+1}}\left(\alpha e^{a_{t+1}}\left(\frac{N_{t+1}}{K_{t+1}}\right)^{1-\alpha} + 1 - \delta\right)\right]$$

Logs

$$-\log C_t = \log\left\{\beta\,\mathrm{E}_t\left[\frac{1}{C_{t+1}}\left(\alpha e^{a_{t+1}}\left(\frac{N_{t+1}}{K_{t+1}}\right)^{1-\alpha}+1-\delta\right)\right]\right\}$$

1st order Taylor series expansion

$$-\log C - \frac{1}{C}(C_t - C) = \log\underbrace{\left\{\beta\left[\frac{1}{C}\left(\alpha\left(\frac{N}{K}\right)^{1-\alpha}+1-\delta\right)\right]\right\}}_{\equiv B}$$

$$+\,\mathrm{E}_t\,\frac{-\beta\left(\alpha\left(\frac{N}{K}\right)^{1-\alpha}+1-\delta\right)}{C^2B}(C_{t+1}-C)$$

$$+\,\mathrm{E}_t\,\frac{\beta\alpha\left(\frac{N}{K}\right)^{1-\alpha}}{CB}(e^{a_{t+1}}-e^0)$$

$$+\,\mathrm{E}_t\,\frac{\beta\alpha(1-\alpha)\left(\frac{N}{K}\right)^{-\alpha}}{CKB}(N_{t+1}-N)$$

$$+\,\mathrm{E}_t\,\frac{-\beta\alpha(1-\alpha)\left(\frac{N}{K}\right)^{-\alpha}N}{K^2CB}(K_{t+1}-K)$$

Constant terms disappear due to steady-state relationship. Divide and multiply each term with the steady-state value of the corresponding variable to obtain

$$-\frac{C}{C}\frac{(C_t-C)}{C} = -\,\mathrm{E}_t\,\frac{C\beta\left(\alpha\left(\frac{N}{K}\right)^{1-\alpha}+1-\delta\right)}{C^2B}\frac{(C_{t+1}-C)}{C}$$

$$+\,\mathrm{E}_t\,\frac{\beta\alpha\left(\frac{N}{K}\right)^{1-\alpha}}{CB}\frac{(e^{a_{t+1}}-e^0)}{1}$$

$$+\,\mathrm{E}_t\,\frac{N\beta\alpha(1-\alpha)\left(\frac{N}{K}\right)^{-\alpha}}{CKB}\frac{(N_{t+1}-N)}{N}$$

$$-\,\mathrm{E}_t\,\frac{K\beta\alpha(1-\alpha)\left(\frac{N}{K}\right)^{-\alpha}N}{K^2CB}\frac{(K_{t+1}-K)}{K}$$

Simplify the coefficients using steady-state relationships and use the "hat" notation to get

$$\begin{aligned}
\hat{C}_t &= \mathrm{E}_t\, \hat{C}_{t+1} \\
&- \mathrm{E}_t\, \beta\alpha \frac{Y}{K} a_{t+1} \\
&- \mathrm{E}_t\, \beta\alpha(1-\alpha)\frac{Y}{K}\hat{N}_{t+1} \\
&+ \mathrm{E}_t\, \beta\alpha(1-\alpha)\frac{Y}{K}\hat{K}_{t+1}
\end{aligned}$$

We log-linearize the other conditions too:

$$\begin{aligned}
\theta\frac{1}{1-N_t} &= \frac{1}{C_t}(1-\alpha)e^{a_t}\left(\frac{K_t}{N_t}\right)^{\alpha} \\
C_t &= e^{a_t}K_t^{\alpha}N_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t
\end{aligned}$$

Logs:

$$\begin{aligned}
&\log\theta - \log(1-N_t) = \log(1-\alpha) - \log C_t + a_t + \alpha\log K_t - \alpha\log N_t \\
&\log C_t = \log\left\{e^{a_t}K_t^{\alpha}N_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t\right\}
\end{aligned}$$

1st order Taylor series expansion

$$\begin{aligned}
&\log\theta - \log(1-N) - \frac{-1}{1-N}(N_t - N) = \log(1-\alpha) - \log C - \frac{1}{C}(C_t - C) \\
&\quad + a_t + \alpha\log K + \frac{\alpha}{K}(K_t - K) - \alpha\log N - \frac{\alpha}{N}(N_t - N) \\
&\log C + \frac{1}{C}(C_t - C) = \log\left\{K^{\alpha}N^{1-\alpha} - \delta K\right\} \\
&\quad + \frac{K^{\alpha}N^{1-\alpha}}{K^{\alpha}N^{1-\alpha} - \delta K}a_t + \frac{\alpha N^{1-\alpha} + (1-\delta)}{K^{\alpha}N^{1-\alpha} - \delta K}(K_t - K) \\
&\quad + \frac{(1-\alpha)K^{\alpha}N^{-\alpha}}{K^{\alpha}N^{1-\alpha} - \delta K}(N_t - N) + \frac{-1}{K^{\alpha}N^{1-\alpha} - \delta K}(K_{t+1} - K)
\end{aligned}$$

Divide and multiply and simplify

$$\frac{N}{1-N}\hat{N}_t = -\hat{C}_t + a_t + \alpha\hat{K}_t - \alpha\hat{N}_t$$

$$\hat{C}_t = +\frac{Y}{Y-\delta K}a_t + \frac{\alpha N^{1-\alpha} + (1-\delta)}{Y/K-\delta}\hat{K}_t$$
$$+\frac{(1-\alpha)Y}{Y-\delta K}\hat{N}_t + \frac{-1}{Y/K-\delta}\hat{K}_{t+1}$$

Hence, our system in log-linearized form

$$\hat{C}_t = \mathrm{E}_t\,\hat{C}_{t+1} - \mathrm{E}_t\,\beta\alpha\frac{Y}{K}a_{t+1} - \mathrm{E}_t\,\beta\alpha(1-\alpha)\frac{Y}{K}\hat{N}_{t+1} + \mathrm{E}_t\,\beta\alpha(1-\alpha)\frac{Y}{K}\hat{K}_{t+1}$$

$$\frac{N}{1-N}\hat{N}_t = -\hat{C}_t + a_t + \alpha\hat{K}_t - \alpha\hat{N}_t$$

$$\hat{C}_t = +\frac{Y}{Y-\delta K}a_t + \frac{\alpha N^{1-\alpha} + (1-\delta)}{Y/K-\delta}\hat{K}_t + \frac{(1-\alpha)Y}{Y-\delta K}\hat{N}_t + \frac{-1}{Y/K-\delta}\hat{K}_{t+1}$$

and

$$a_t = \rho a_{t-1} + \varepsilon_t.$$

Step 4. Solving the model

Since our model is in linear form, we may solve the model using linear techniques

- Undetermined coefficients, Uhlig (1997)
- Blanchard and Kahn (1980)
- Schur decomposition, Sims (200?), Klein (200?)
- ...

Method of undetermined coefficients

Following Anderson and Moore (1985) (AiM) and Zagaglia (2005), DSGE model may be written in the form

$$H_{-1}z_{t-1} + H_0 z_t + H_1\,\mathrm{E}_t\,z_{t+1} = D\eta_t, \tag{7.13}$$

where z_t is vector of endogenous variables and η_t are pure innovations with zero mean and unit variance.

The solution to (7.13) takes the form

$$z_t = B_1 z_{t-1} + B_0 \eta_t,$$

where

$$\begin{aligned} B_0 &= S_0^{-1} D, \\ S_0 &= H_0 + H_1 B_1. \end{aligned}$$

B_1 satisfies the identity

$$H_{-1} + H_0 B_1 + H_1 B_1^2 = 0.$$

This is a quadratic equation on the unknown B_1. One of the (set of) roots, will stable (less than unity in absolute values) and one unstable. We need to pick the stable one!

We solve this numerically by computer.

Defining z_t, η_t and H_{-1}, H_0, H_1, D in our case:

- We may substitute C_t in our system of equations and express everything in terms of $N_t, N_{t+1}, K_t, K_{t+1}, a_t, a_{t-1}$.
- Then

$$z_t \equiv \begin{bmatrix} K_{t+1} \\ N_t \\ a_t \end{bmatrix}. \text{ and } \eta_t \equiv \varepsilon_t.$$

- Matrices H_{-1}, H_0, H_1, D are stacked. (not listed here)

Calibration

- To solve the model numerically we need to assign values for the parameters.
- These may come from many sources:
  - macroeconomic data
  - microeconometric studies
  - matching model moments with data moments: eg relative variances, autocorrelation, cross-correlations, etc.
- Our calibration

Parameter	β	θ	α	δ	ρ	σ
Value	0.99	1.75	0.33	0.023	0.95	0.01

Summarizing numerical analysis

Summarizing model

- We need to summarize model features: steady-states, means, variances, autocorrelations
- Impulse responses:
  - Summarizing dynamics
  - In period $t = 1$ set shock innovations to some value (eg standard error) and to zero after this

$$\varepsilon_1 = 0.01 \text{ and } \varepsilon_t = 0 \quad t = 2, 3, \ldots$$

calculate the values of all other variables in periods $t = 1, 2, \ldots, T$

STEADY-STATE RESULTS:

y	1.0301
c	0.793902
k	10.2696
i	0.236201
n	0.331892
y_l	3.10373
a	0
r	0.033101
w	2.0795

VARIABLE	MEAN	STD. DEV.	VARIANCE
y	1.0301	0.0140	0.0002
c	0.7939	0.0034	0.0000
k	10.2696	0.0387	0.0015
i	0.2362	0.0110	0.0001
n	0.3319	0.0022	0.0000
y_l	3.1037	0.0223	0.0005
a	0.0000	0.0136	0.0002
r	0.0331	0.0004	0.0000
w	2.0795	0.0149	0.0002

POLICY AND TRANSITION FUNCTIONS

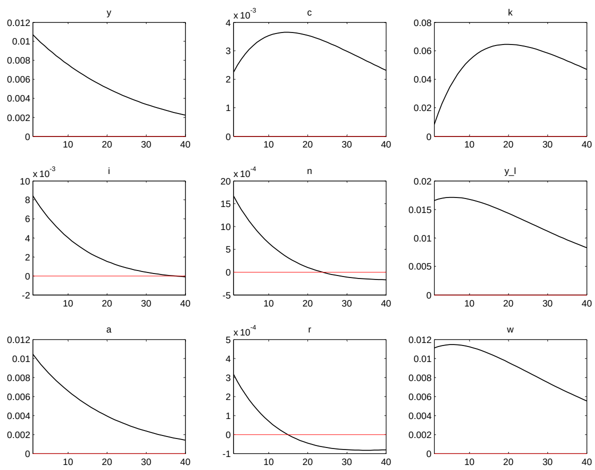
	y	c	k	i	n	y_l	a
Constant	1.030103	0.793902	10.269592	0.236201	0.331892	3.103727	0
k(-1)	0.016113	0.041667	0.951446	-0.025554	-0.008169	0.124945	0
a(-1)	0.971376	0.205065	0.766311	0.766311	0.151823	1.506992	0.950000
e	1.022501	0.215858	0.806643	0.806643	0.159814	1.586308	1.000000

MATRIX OF CORRELATIONS (HP filter, lambda = 1600)

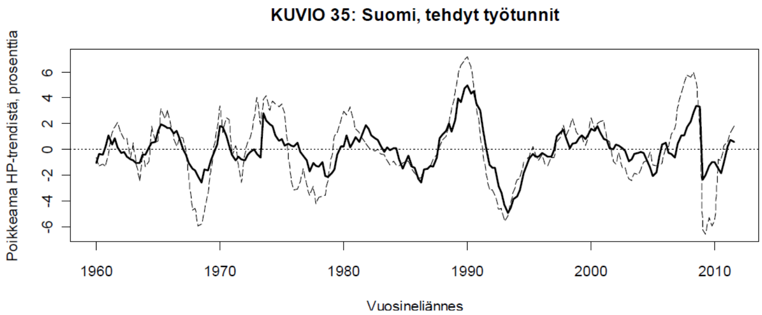
Variables	y	c	k	i	n	y_l	a	r	w
y	1.0000	0.9001	0.3510	0.9910	0.9821	0.9849	0.9990	0.9608	0.9
c	0.9001	1.0000	0.7238	0.8335	0.8021	0.9620	0.8798	0.7441	0.9
k	0.3510	0.7238	1.0000	0.2222	0.1685	0.5079	0.3088	0.0777	0.5
i	0.9910	0.8335	0.2222	1.0000	0.9985	0.9527	0.9960	0.9893	0.9
n	0.9821	0.8021	0.1685	0.9985	1.0000	0.9347	0.9896	0.9958	0.9
y_l	0.9849	0.9620	0.5079	0.9527	0.9347	1.0000	0.9762	0.8983	1.0
a	0.9990	0.8798	0.3088	0.9960	0.9896	0.9762	1.0000	0.9722	0.9
r	0.9608	0.7441	0.0777	0.9893	0.9958	0.8983	0.9722	1.0000	0.8
w	0.9849	0.9620	0.5079	0.9527	0.9347	1.0000	0.9762	0.8983	1.0

COEFFICIENTS OF AUTOCORRELATION (HP filter, lambda = 1600)

Order	1	2	3	4	5
y	0.7176	0.4780	0.2790	0.1178	-0.0090
c	0.8043	0.6152	0.4385	0.2783	0.1373
k	0.9588	0.8596	0.7225	0.5643	0.3981
i	0.7075	0.4621	0.2605	0.0992	-0.0260
n	0.7060	0.4597	0.2578	0.0965	-0.0285
y_l	0.7430	0.5182	0.3257	0.1648	0.0338
a	0.7133	0.4711	0.2711	0.1098	-0.0163
r	0.7070	0.4612	0.2596	0.0982	-0.0269
w	0.7430	0.5182	0.3257	0.1648	0.0338



Stylized facts of labour markets



Volatility of hours is a bit less than than the output. Also *pro-cyclical* (may be lagging a bit)

Discussion

Basic mechanism: intertemporal substitution ad capital accumulation!
Basic results:

- Work hard when productivity is high: "make the hay while sun shines"
- You want to save when productivity is high: consumption smoothing

Comparison with the US economy:

- Simulated economy's output fluctuatios are around 75 % of the observed US fluctuations.
- Consumption is less volatile than output
- Investment is much more volatile than output
- Hours

Assesment

- Explains substantial part of fluctuations
- Explains correlation of many variables
- Problems in explaining hours (not enough volatility)
- No unemployment
- Where does the productivity a_t shocks come from

- We should observe negative shocks $\Longleftrightarrow$ technological regressions!
- No heterogeneity
- *No monetary non-neutrality*
  $\longrightarrow$Will be fixed in the next section.

Policy implications

- Pareto efficient
- Fluctuations are agents' optimal response to changes in their environment
- These are pareto optimal!
  $\longrightarrow$Policy action cannot improve the outcome!
  $\longrightarrow$Policy can only make things *worse!*
- Recessions have a 'cleansing' effect.

Extensions are many, among them

- Fiscal policy shocks (McGrattan 1994)
- Indivisible labour (Rogerson 1988, and Hansen 1985)
- Home production (Benhabib, Rogerson and Wright, 1991)
- Money (Cooley and Hansen 1989)
- Finite lives (Rios-Rull, 1996)

8 New-Keynesian model and monetary policy

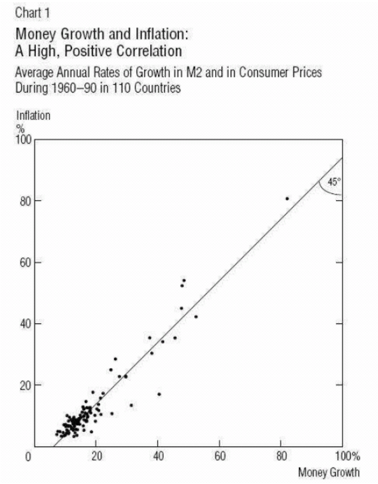
8.1 Introduction

Empirical evidence on money, inflation and output

Help to judge theoretical models: theoretical models should be consistent with the empirical data

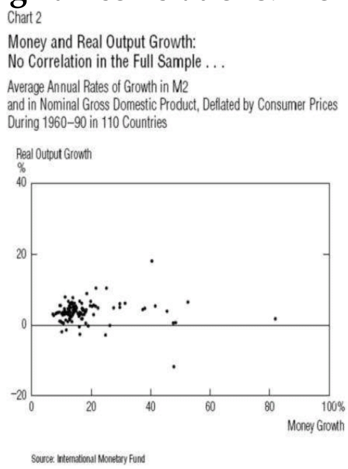
Help to evaluate the effects of money and monetary policy

Long-run correlations: money growth and inflation



- long-run correlations between money growth and inflation close to one
- quantitative theory of money predicts: $\Delta m \longrightarrow \pi$. *Causality?*

Long-run correlations: money growth and output growth



No long-run trade-off between inflation and unemployment
$\longrightarrow$Phillips curve vertical (in the long-run).

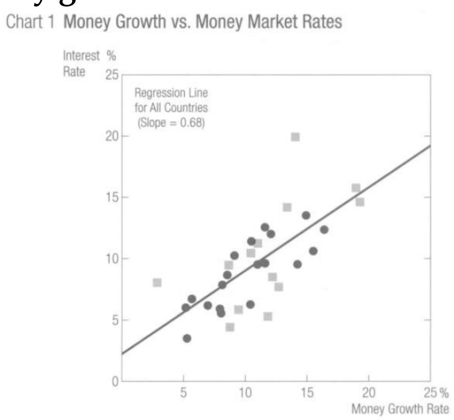
Fisher equation

Fisher equation: $i_t = r_t + \mathrm{E}_t\, \pi_{t+1}$
$\longrightarrow i_{SS} = r_{SS} + \pi_{SS}$

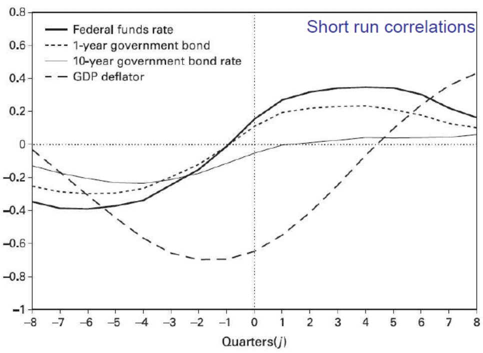
- Fisher equation: nominal interest rates should be positively related to expected inflation, suggesting that the level of nominal interest rate positively link to average inflation in the long run. Then nominal interest rates and average money growth rates should also be positively correlated

- Monnet and Weber (2001): higher correlation between money growth and long-term interest rates. Consistent with Fisher equation
- Mishkin (1992): test for the presence of a long-run relationship between the interest rate and money growth rate. Support Fisher equation in the long run (trend together)

Money growth and interest rates



Short-run correlations: interest rates, inflation, output



Interest rates are negatively correlated with GDP at lags and positively at leads

Prices are negatively correlated with GDP at lags (and contemporaneously) and positively at (long) leads

Cross-section evidence

Supportive to price rigidity

4–6 quarters by Bils and Klenow (2004), 8–11 quarters by Nakamura and Steinsson (2008), 8-11 by Álvarez et al. (2006); Dhyne et al. (2006).

Wages are also rigid: 1 year, downward rigidity by Dickens et al. (2007)

New Keynesian Model

- Methodologically similar to RBC models.
- Builds on the following features:
  - monopolistic competition
  - nominal rigidities
  - → short-run non-neutrality of money: real interest rate affect money supply.
- Leads to differences w.r.t RBC models: economy's response to shocks is generally *inefficient*.
- Removing the effects of non-neutrality is potentially welfare improving → role for monetary policy.

8.2 The model with no price stickiness

Many details and part of the discussions relies on lecture notes by Eric Sims.

8.2.1 Households

Households

Household problem is similar to setups we have already studied.
We assume

- Constant elasticity of substitution preferences over consumption C_t
- Supply labour N_t.
- One-period *nominal* bond B_t (end-of-period) with nominal interest rate i_t.
- Note that there is no *physical capital* in this model. This is probably not innocent assumption but simplifies algebra substantially. The nominal bond serves as the savings device.

- Price level P_t.
- Nominal wages (per hour) W_t
- Profits D_t.
- Households choose $\{C_t, B_t, N_t\}_{t=0}^{\infty}$ taking i_t, W_t, P_t as given.

  Maximization problem

$$\max_{\{C_t,B_t,N_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} + \theta \frac{(1-N_t)^{1-\xi}-1}{1-\xi} \right]$$

subject to

$$P_t C_t + B_t = W_t N_t + D_t + (1+i_{t-1}) B_{t-1}.$$

The Lagrangian can be written as

$$\mathcal{L} = \mathrm{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} + \theta \frac{(1-N_t)^{1-\xi}-1}{1-\xi} \right] \right.$$
$$\left. + \lambda_t \left[W_t N_t + D_t + (1+i_{t-1}) B_{t-1} - P_t C_t - B_t \right] \right\}$$

The first-order conditions

$$\begin{aligned} C_t: &\quad \beta^t C_t^{-\sigma} = \lambda_t P_t \\ N_t: &\quad \beta^t \theta (1-N_t)^{-\xi} = \lambda_t W_t \\ B_t: &\quad \lambda_t = \mathrm{E}_t \, \lambda_{t+1} (1+i_t). \end{aligned}$$

These simplify to

$$\theta (1-N_t)^{-\xi} = C_t^{-\sigma} \frac{W_t}{P_t}$$
$$C_t^{-\sigma} = \beta \, \mathrm{E}_t \, C_{t+1}^{-\sigma} \frac{(1+i_t)}{\Pi_{t+1}},$$

where $\Pi_t \equiv P_t / P_{t-1}$ is *gross* inflation $1 + \pi_t$

8.2.2 Production

Production

Production happens in two stages

Final goods
Final goods in a constant elasticity of substitution (CES) aggregate of continuum of intermediate goods.

Intermediate goods
Each intermediate good is produced by the same production function using labour. Level of technology is also the same for each individual firm.

Profit maximization of in the final goods sector yields a downward-sloping demand curve for intermediate goods. Since the output of intermediate goods are imperfectly substitutable
$\longrightarrow$imperfect (=monopolistic) competition
$\longrightarrow$pricing power!

What differentiates monopolistic competition from perfect competition is that large number of firms are selling differentiated products and have some pricing power. Due to entry and exit, no profits exist in the long run.

Final goods
The is one final goods firm and a continuum (infinite amount) of intermediate goods firms. The intermediate goods firms are indexed along the unit interval.

The "production function" of the final good firm is

$$Y_t = \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}},$$

where $\epsilon > 1$ is the elasticity of substitution. For any $\epsilon < \infty$ the goods are imperfect substitutes. This will result market power for producers of $Y_t(j)$.

The maximization problem of final goods firm is

$$\max_{Y_t(j)} P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

subject to the production function

$$Y_t = \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}}.$$

Substitute the production function into profit equation (the first row):

$$\max_{Y_t(j)} P_t \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

The first order condition (differentiating the above profits w.r.t. $Y_t(j)$ is the following

$$P_t \frac{\epsilon}{\epsilon-1}\left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}dj\right]^{\frac{\epsilon}{\epsilon-1}-1}\frac{\epsilon-1}{\epsilon}Y_t(j)^{\frac{\epsilon-1}{\epsilon}-1}=P_t(j) \quad \forall j$$

Simplify

$$\begin{aligned}
&P_t\left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}dj\right]^{\frac{\epsilon}{\epsilon-1}-\frac{\epsilon-1}{\epsilon-1}} Y_t(j)^{\frac{\epsilon-1}{\epsilon}-\frac{\epsilon}{\epsilon}}=P_t(j) \quad \forall j\\
&P_t\left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}dj\right]^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}}=P_t(j) \quad \forall j\\
&Y_t(j)^{-\frac{1}{\epsilon}}=\left(\frac{P_t(j)}{P_t}\right)\left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}dj\right]^{-\frac{1}{\epsilon-1}} \quad \forall j\\
&Y_t(j)=\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}\left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}dj\right]^{\frac{\epsilon}{\epsilon-1}} \quad \forall j\\
&Y_t(j)=\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t \quad \forall j
\end{aligned} \tag{8.1}$$

This says

- Demand for good $Y_t(j)$ will depend *negatively* on its relative price $\frac{P_t(j)}{P_t}$ and the elasticity of substitution ϵ.
- and *positively* from the aggregate production Y_t.
- if $\epsilon \to \infty$ the demand becomes infinitely elastic $\longrightarrow$perfect competition!

Since the final goods producer is competitive, the profits are zero

$$P_tY_t=\int_0^1 P_t(j)Y_t(j)dj$$

Substitute the above demand function into this

$$P_tY_t=\int_0^1 P_t(j)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}Y_tdj$$

Since Y_t and P_t does not depend on j, we may "take them out from integral":

$$P_t Y_t = P_t^{\epsilon} Y_t \int_0^1 P_t(j)^{1-\epsilon} dj$$
$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$
$$P_t = \left[\int_0^1 P_t(j)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}}.$$

This defines the aggregate price index.

Intermediate goods

There is an infinite number of intermediate goods producers populated along the unit interval.

They use labour as a factor of production and shares the same total factor productivity (TFP), ie the level of technology. Production function takes the linear form

$$Y_t(j) = A_t N_t(j).$$

Since they face a downward-sloping demand curve given by (8.1), they must in addition to employment/hours choose the price of the production. We do the optimization in two steps.

Determination of employment/hours

- Intermediate goods producers are price-takers in the factor markets $\longrightarrow$they take the *common* (nominal) wages W_t as given.
- They produce as much output $Y_t(j)$ as is demanded at a given price $P_t(j)$.
- Their optimization problem is static $\longrightarrow$period by period maximization $\Longleftrightarrow$ cost minimization

$$\min_{N_t(j)} W_t N_t(j)$$

subject to

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$$
$$Y_t(j) = A_t N_t(j).$$

- Profits are maximized when the costs are minimized. Production equals the amount of demanded.
- Lagrangian

$$\mathcal{L} = -W_t N_t(j) + \varphi_t \left[A_t N_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \right]$$

and the first order condition

$$W_t = \varphi_t A_t.$$

- We may interpret the Lagrange multiplier φ_t as the *nominal marginal cost*, ie how much nominal costs change (the function to be minimized) if the constraint is relaxed (if one more unit is produced).
- The marginal cost is the same for each firm due to same technology, competitive factor markets and constant-returns-to-scale production function.
- Nominal marginal costs are then given by

$$\varphi_t = \frac{W_t}{A_t}$$

and *real* marginal costs (dividing both sides by P_t)

$$\frac{\varphi_t}{P_t} = \frac{W_t/P_t}{A_t}.$$

Note that in the case of perfect competition, the price level would equal the marginal costs, ie $P_t = \varphi_t$ and real marginal costs would be *unity!*

- The labour demand is given by the production function

$$N_t(j) = \frac{Y(j)}{A_t}.$$

Price determination

- Optimal choice of price when labour choice is optimal

- Firm maximizes its profits by choosing the price

$$\max_{P_t(j)} P_t(j)Y_t(j) - W_t N_t(j)$$

subject to

$$\begin{aligned} Y_t(j) &= \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t \\ W_t &= \varphi_t A_t. \end{aligned}$$

Substitute the constraints into the objective function

$$\begin{aligned} \max_{P_t(j)} P_t(j) \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t - \varphi_t A_t N_t(j) \\ = P_t(j) \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t - \varphi_t \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t \end{aligned}$$

- Firm is small
⟶Takes the *aggregate* price level P_t and output Y_t as given.

- The first order conditions

$$(1-\epsilon)P_t(j)^{-\epsilon}P_t^{-\epsilon}Y_t + \epsilon\varphi_t P_t(j)^{-\epsilon-1}P_t^{-\epsilon}Y_t = 0.$$

that simplify to

$$P_t(j) = \frac{\epsilon}{\epsilon - 1}\varphi_t \quad \forall j.$$

- This tells
  - Markup is given by
  $$\mathcal{M} \equiv \frac{\epsilon}{\epsilon - 1}$$
  Since $\epsilon > 1$, $\mathcal{M} > 1$.
  - Price of good j equals markup over the (nominal) marginal costs.
  - Price is higher than the marginal cost.
  - The less substitutability (the more "different" the goods are), ie the close ϵ is to unity, the higher is the markup $\mathcal{M}$.
  - When approaching to full competition $\epsilon \to \infty$, then $\mathcal{M} \to 1$, and $P_t(j) \to \varphi_t$.

8.2.3 Aggregation

Aggregation

All firms behave identically = symmetric equilibrium:

- Competitive factor markets
  $\longrightarrow$same marginal costs φ_t.
- Same demand elasticity
  $\longrightarrow$all choose the same price
  $\longrightarrow$all face the same demand
  $\longrightarrow$all produce the same amount
  $\longrightarrow$all hire same amount of labour (due to the same A_t)

Since the each firm produce the same amount, the aggregate production function implies

$$Y_t = \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}} = Y_t(j)\left[\int_0^1 dj\right]^{\frac{\epsilon}{\epsilon-1}} = Y_t(j)$$

Output of the final good is the same as output of the intermediate goods. Since we are summing over unit interval, the sum is equal to the amount produced by any one firm on the unit interval. Hence

$$Y_t(j) = \int_0^1 Y_t(j)dj.$$

Also

$$Y_t = Y_t(j) = A_t N_t(j) = \int_0^1 A_t N_t(j)dj = A_t \int_0^1 N_t(j)dj = A_t N_t.$$

Due to

$$N_t = \int_0^1 N_t(j)dj.$$

Since all intermediate goods producers behave the same, the aggregate price level is the same as the price level of any intermediate goods firm

$$P_t = P_t(j) = \underbrace{\frac{\epsilon}{\epsilon-1}}_{=\mathcal{M}>1}\varphi_t.$$

The labour demand condition implies

$$W_t = \varphi_t A_t.$$

and real wages $w_t \equiv W_t / P_t$

$$w_t = \frac{\varphi_t}{P_t} A_t.$$

and due to pricing condition

$$w_t = \underbrace{\frac{\epsilon - 1}{\epsilon}}_{=\mathcal{M}^{-1}<1} \underbrace{A_t}_{=\frac{\partial Y_t(j)}{\partial N_t(j)}}$$

ie less than the marginal product of labour! (Due to markup.)

Due to the markup $\mathcal{M} > 1$

- Prices are higher that in a competitive economy.
- Real wages are lower than in a competitive economy

Equilibrium

In *equilibrium*

- Labour markets clear

$$N_t = \int_0^1 N_t(j) dj.$$

- Bond-holdings is always zero, $B_t = 0$.
- Using above, the household budget constraint is given by

$$P_t C_t = W_t N_t + D_t$$

The final goods producer operates under perfect competition. Hence, its profits are zero.

The profits of the intermediate goods producer is given by

$$D_t = \int_0^1 \left(P_t(j) Y_t(j) - W_t N_t(j)\right) dj$$
$$= \int_0^1 P_t(j) Y_t(j) dj - W_t \int_0^1 N_t(j) dj$$

Due to the labour market clearing this is

$$D_t = \int_0^1 P_t(j) Y_t(j) dj - W_t N_t.$$

Substituting this back to budget constraint gives

$$P_t C_t = \int_0^1 P_t(j) Y_t(j) dj.$$

Finally, due to symmetry $P_t = P_t(i)$, $Y_t(j) = Y_t$, the above results

$$C_t = Y_t$$

The system can be summarized as

$$\theta(1-N_t)^{-\xi} = C_t^{-\sigma} \frac{W_t}{P_t}$$
$$C_t^{-\sigma} = \beta \, \mathrm{E}_t \, C_{t+1}^{-\sigma} \frac{(1+i_t)}{\Pi_{t+1}}$$
$$C_t = Y_t$$
$$Y_t = A_t N_t$$
$$W_t = P_t \underbrace{\frac{\epsilon - 1}{\epsilon}}_{=\mathcal{M}^{-1}} A_t$$
$$\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0, \sigma^2).$$

where $\Pi_t = P_t / P_{t-1}$.

- The first one describes the labour supply.
- The second one is the consumption Euler equation
- Then the goods market condition, ie aggregate budget constraint
- and production function
- and labour demand
- finally, the equation of motion of the technology.

The linearized version of the model is the following

$$\xi \hat{N}_t = -\log \theta - \sigma \hat{C}_t + \hat{W}_t - \hat{P}_t \tag{8.2}$$
$$\hat{C}_t = \mathrm{E}_t \, \hat{C}_{t+1} - \frac{1}{\sigma} (i_t - \mathrm{E}_t \, \pi_{t+1} + \log \beta) \tag{8.3}$$
$$\hat{C}_t = \hat{Y}_t \tag{8.4}$$
$$\hat{Y}_t = a_t + \hat{N}_t \tag{8.5}$$
$$\hat{W}_t = \hat{P}_t + \log \mathcal{M}^{-1} + a_t \tag{8.6}$$
$$a_t = \rho a_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0, \sigma^2) \tag{8.7}$$

where $\mathcal{M} \equiv \epsilon/(\epsilon - 1)$.

Substitute (8.5) into (8.4) and then to (8.2) with (8.6) to get

$$\xi \hat{N}_t = -\log\theta - \sigma(a_t + \hat{N}_t) + \log\mathcal{M}^{-1} + a_t$$

that gives

$$\hat{N}_t = \frac{1}{\mathcal{M}\theta(\xi + \sigma)} + \frac{1 - \sigma}{\xi + \sigma} a_t$$

and

$$\hat{Y}_t = \frac{1}{\mathcal{M}\theta(\xi + \sigma)} + \left(\frac{1 + \xi}{\xi + \sigma}\right) a_t$$

Define real interest rate (according to Fisher equation):

$$r_t \equiv i_t - \mathrm{E}_t\, \pi_{t+1}$$

and $\bar{r} \equiv -\log\beta$. Then

$$\begin{aligned}
\hat{C}_t &= \mathrm{E}_t\, \hat{C}_{t+1} - \frac{1}{\sigma}(r_t - \bar{r}) \\
\hat{Y}_t &= \mathrm{E}_t\, \hat{Y}_{t+1} - \frac{1}{\sigma}(r_t - \bar{r}) \\
r_t &= \bar{r} + \sigma\left(\mathrm{E}_t\, \hat{Y}_{t+1} - \hat{Y}_t\right) \\
&= \bar{r} + \sigma\left(\frac{1 + \xi}{\xi + \sigma}\right) \mathrm{E}_t(a_{t+1} - a_t).
\end{aligned}$$

Notes:

- Output response is positive.
- Employment response depends on $1 - \sigma$.
  - If $\sigma > 1$ the response is negative. Income effect dominates!
  - If $\sigma < 1$ the response is positive. Substitution effect dominates!
  - If $\sigma = 1$ the response is zero. Income and substitution effect cancel each other. Logarithmic utility!
- Real interest rate depends on the expected growth rate of technology.
- Output and employment are independent of monetary policy!

Monetary policy in the flexible price model
Fixed nominal interest rate
Consider standard Fisher equation

$$i_t = \mathrm{E}_t\, \pi_{t+1} + r_t = \mathrm{E}_t\, p_{t+1} - p_t + r_t. \tag{8.8}$$

Iterate forward to get

$$p_t = -\,\mathrm{E}_t \sum_{i=0}^{\infty} (i_{t+i} - r_{t+i}).$$

It is easy to see that this does not converge for any fixed interest rate $i_{t+i} = i$.
Simple inflation based interest rate rule Consider the following *Taylor rule*:

$$1 + i_t = \frac{1}{\beta} \Pi_t^{\phi_\pi}.$$

This is easier to interpret if we take the logs of both sides

$$\begin{aligned} &\log(1 + i_t) = -\log\beta + \phi_\pi \log \Pi_t \quad <\approx> \\ &i_t = \bar{r} + \phi_\pi \pi_t, \end{aligned}$$

where $\bar{r} = -\log(\beta)$ is the rate of the time preference and π_t is the inflation rate.
Combine it with the Fisher equation to obtain

$$\bar{r} + \phi_\pi \pi_t = \mathrm{E}_t\, \pi_{t+1} + r_t.$$

or

$$\pi_t = \frac{1}{\phi_\pi} \left(\mathrm{E}_t\, \pi_{t+1} + (r_t - \bar{r}) \right)$$

Iterate it forward to obtain

$$\pi_t = \frac{1}{\phi_\pi} \mathrm{E}_t \sum_{i=0}^{\infty} \left(\frac{1}{\phi_\pi} \right)^i (r_{t+i} - \bar{r}).$$

Taylor principle
The inflation will be finite if and only if

$$\phi_\pi > 1$$

8.3 Model with price rigidities

8.3.1 Calvo pricing

Introduction

In the above, firms were able to change their price in every period.

The intermediate goods producers set their price as a constant mark-up over the marginal costs.

In the following, *the firms can only change their price in a certain period with a probability of* $1 - \phi$.

Side-step: the firm value

Remember the optimality condition for share prices in the household problem of the decentralized economy:

$$q_t = \beta \, \mathrm{E}_t \, \frac{C_t}{C_{t+1}} \left(q_{t+1} + D_t \right),$$

where q_t is the equity price and D_t is the profits. It is easy the check that the same equation with the utility function of this chapter would write

$$q_t = \beta \, \mathrm{E}_t \, \frac{C_t^\sigma}{C_{t+1}^\sigma} \left(q_{t+1} + D_t \right)$$

and for k period holdings

$$q_t = \beta^k \, \mathrm{E}_t \, \frac{C_t^\sigma}{C_{t+k}^\sigma} \left(q_{t+k} + D_t \right).$$

The term

$$\Lambda_{t,t+k} \equiv \beta^k \, \mathrm{E}_t \, \frac{C_t^\sigma}{C_{t+k}^\sigma}$$

is often called k period *stochastic discount factor*. It tells how we price any claims in this economy. (It forms the essence of macroeconomic theory of asset prices.) *This is the discount factor one should use when evaluating the profits (shares) of a firm.*

Calvo Fairy

A firm may change price of its product only when Calvo Fairy visits.

The probability of a visit is $1 - \phi$.

It is independent of the length of the time and the time elapsed since the last adjustment. Hence, in each period the $1 - \phi$ share of firms may change their price and rest, ϕ, keep their price unchanged.

Mathematically, Calvo Fairy's visits follows Bernoulli process (discrete version of Poisson process).

The probability distribution of the number of periods between the visits of Calvo Fairy is geometric distribution.

The expected value of geometric distribution and, hence, the average number of periods between the price changes (of a firm) is

$$\frac{1}{1-\phi}.$$

Optimal price setting

Let $P_t^\star$ denote the price level of the firm that *receives price change signal*. This is the price level of the firm that Calvo Fairy visits.

The momentary profits of this firm that chooses the price $P_t^\star$ are given (as above)

$$D_t(P_t^\star) = P_t^\star Y_t(j) - W_t N_t(j)$$

and the constraints (again as above)

$$\begin{aligned} Y_t(j) &= \left(\frac{P_t^\star}{P_t}\right)^{-\epsilon} Y_t \\ W_t &= \varphi_t A_t. \end{aligned}$$

Plugging the constraints to the profits, gives

$$\begin{aligned} &P_t^\star \left(\frac{P_t^\star}{P_t}\right)^{-\epsilon} Y_t - \varphi_t A_t N_t(j) \\ &\qquad = P_t^\star \left(\frac{P_t^\star}{P_t}\right)^{-\epsilon} Y_t - \varphi_t Y_t(j) \\ &\qquad\qquad = P_t^\star \left(\frac{P_t^\star}{P_t}\right)^{-\epsilon} Y_t - \varphi_t \left(\frac{P_t^\star}{P_t}\right)^{-\epsilon} Y_t \end{aligned}$$

The firm is stuck with $P_t^\star$ with probability ϕ

What is important here is that the firm is prepared that it is stuck with the price level $P_t^\star$ *if it cannot change this price in the future*. This is why $P_t^\star$ does not depend on k.

Today it is stuck with probability 1.

Tomorrow with probability ϕ,

Day after tomorrow with probability ϕ^2,

$\vdots$

After k periods with probability ϕ^k.

When making its pricing decision, the firm takes into account that it can change its price with the probability $1-\phi$, ie the chosen price remains the same with probability ϕ. The expected discounted profits are given by

$$\max_{P_t^\star} \mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left\{ P_t^\star \left(\frac{P_t^\star}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} - \varphi_{t+k} \left(\frac{P_t^\star}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \right\},$$

where $\Lambda_{t,t+k}$ is defined as above.

The first order condition of the above maximization problem is given by

$$\mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[(1-\epsilon)(P_t^\star)^{-\epsilon} P_{t+k}^{-\epsilon} Y_{t+k} - (-\epsilon)(P_t^\star)^{-1-\epsilon} P_{t+k}^{-\epsilon} \varphi_{t+k} Y_{t+k} \right] = 0.$$

Let's simplify

$$\mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[(\epsilon-1)(P_t^\star)^{-\epsilon} P_{t+k}^{-\epsilon} Y_{t+k} \right] = \mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[\epsilon (P_t^\star)^{-1-\epsilon} P_{t+k}^{-\epsilon} \varphi_{t+k} Y_{t+k} \right]$$

and since the price they choose in period t does not depend on k, we may pull these out from the infinite sums

$$(\epsilon-1)(P_t^\star)^{-\epsilon} \mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[P_{t+k}^{-\epsilon} Y_{t+k} \right] = \epsilon (P_t^\star)^{-1-\epsilon} \mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[P_{t+k}^{-\epsilon} \varphi_{t+k} Y_{t+k} \right]$$

and solve $P_t^\star$

$$P_t^\star = \frac{\epsilon}{\epsilon-1} \frac{\mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[P_{t+k}^{-\epsilon} \varphi_{t+k} Y_{t+k} \right]}{\mathrm{E}_t \sum_{k=0}^{\infty} \phi^k \Lambda_{t,t+k} \left[P_{t+k}^{-\epsilon} Y_{t+k} \right]}.$$

Note:

- Following the reasoning in the above "Aggregation" subsection, the firms face the same marginal cost,

- and take the aggregate variables as given (remember that they are infinitely small)
  $\longrightarrow$any firm that can change its price (due to visit of Calvo fairy) sets the same price!
- The current price that such a firm chooses is the present discounted (now with $\phi^k \Lambda_{t,t+k}$) value of *marginal costs*.
- As above, if $\phi = 0$ (no price rigidities), then

$$P_t^\star = \frac{\epsilon}{\epsilon - 1}\varphi_t.$$

Define *real marginal costs*

$$MC_t \equiv \frac{\varphi_t}{P_t} \text{ then } \frac{P_t^\star}{P_t} = \frac{\epsilon}{\epsilon - 1} MC_t = \mathcal{M} MC_t.$$

The optimal price can be log-linearized (loooonnnggg sequence of steps) resulting

$$\hat{P}_t^\star = \log \mathcal{M} + (1 - \beta\phi)\, \mathrm{E}_t \sum_{k=0}^{\infty} (\phi\beta)^k (\widehat{MC}_{t+k} + \hat{P}_{t+k}).$$

This can be reshuffled (after few pages of boring algebra) to the recursive form

$$\hat{P}_t^\star - \hat{P}_t = \phi\beta\, \mathrm{E}_t(\hat{P}_{t+1}^\star - \hat{P}_{t+1}) + \phi\beta\, \mathrm{E}_t\, \hat{\Pi}_{t+1} + (1 + \phi\beta)\widehat{MC}_t. \tag{8.9}$$

Aggregate price level

Next issue is to study how aggregate price level evolves over time. Consider our price index

$$P_t = \left[\int_0^1 P_t(j)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}}.$$

Remember that the Calvo fairy visit with the same probability in each firm. Now, in each period, the fraction of the firms that can change their price is $1 - \phi$, the rest ϕ firms cannot reset their price. The price level of the latter firms will equal the previous aggregate price level:

$$\begin{aligned} P_t &= \left[\int_0^1 \left((1-\phi)(P_t^\star)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}\right) dj\right]^{\frac{1}{1-\epsilon}} \\ &= \left[\int_0^{1-\phi} (P_t^\star)^{1-\epsilon} dj + \int_{1-\phi}^1 P_{t-1}^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}} \\ &= \left[(1-\phi)(P_t^\star)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \end{aligned}$$

The linearized version of this is

$$\hat{P}_t^\star - \hat{P}_t = \frac{\phi}{1-\phi}\hat{\Pi}_t.$$

Substituting (8.9) to the above equation results

$$\hat{\Pi}_t = \beta \, \mathrm{E}_t \, \hat{\Pi}_{t+1} + \frac{(1-\phi)(1-\beta\phi)}{\phi}\widehat{MC}_t \tag{8.10}$$

This is called the *New-Keynesian Phillips Curve*. It says that inflation is the expected discounted value of future marginal costs.

We are ready to define the equilibrium. It follows the same logic as in the previous subsection:

- Labour markets clear

  $$N_t = \int_0^1 N_t(j)dj.$$

- Bond-holdings is always zero, $B_t = 0$.
- Using above, the household budget constraint is given by

  $$P_t C_t = W_t N_t + D_t$$

  The final goods producer operates under perfect competition. Hence, its profits are zero.

  The profits of the intermediate goods producer is given by

  $$\begin{aligned} D_t = \int_0^1 \left(P_t(j)Y_t(j) - W_t N_t(j)\right) dj \\ = \int_0^1 P_t(j)Y_t(j)dj - W_t \int_0^1 N_t(j)dj \end{aligned}$$

  Due to the labour market clearing this is

  $$D_t = \int_0^1 P_t(j)Y_t(j)dj - W_t N_t.$$

  Substituting this back to the above budget constraint gives

  $$P_t C_t = \int_0^1 P_t(j)Y_t(j)dj$$

Plug the demand function into this

$$P_tC_t = \int_0^1 P_t(j)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t dj$$

$$= Y_t P_t^{\epsilon} \int_0^1 P_t(j)^{1-\epsilon} dj$$

Since the aggregate price index gives

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$

the above simplifies to

$$C_t = Y_t.$$

Hence, goods markets clear at the aggregate/economy-wide level.

- Next we find the aggregate output Y_t. The demand for variety/good j is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$$

and using production function it gives

$$A_t N_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$$

integrate both sides of the equation over j

$$\int_0^1 A_t N_t(j) dj = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t dj$$

which simplifies to

$$A_t \int_0^1 N_t(j) dj = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj$$

Using the labour market clearing condition gives

$$A_t N_t = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj$$

Hence

$$Y_t = \frac{A_t N_t}{\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj}.$$

The term in the denominator is *price dispersion*.

– If there were no *price frictions*, all the firms would charge the same price and the dispersion would be unity!

– If there are price frictions, this term is above or equal to unity $\longrightarrow$The aggregate output is *below that of the frictionless economy*!

To summarize

$$\theta(1-N_t)^{-\xi} = C_t^{-\sigma}\frac{W_t}{P_t}$$

$$C_t^{-\sigma} = \beta\,\mathrm{E}_t\,C_{t+1}^{-\sigma}\frac{(1+i_t)}{\hat{\Pi}_{t+1}}$$

$$C_t = Y_t, \quad Y_t = \frac{A_t N_t}{\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj}$$

$$MC_t \equiv \frac{\varphi_t}{P_t}, \quad W_t = P_t MC_t A_t$$

$$P_t = \left[(1-\phi)(P_t^\star)^{1-\epsilon} + \phi P_t^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$$

$$P_t^\star = \frac{\epsilon}{\epsilon-1}\frac{\mathrm{E}_t\sum_{k=0}^{\infty}\phi^k\Lambda_{t,t+k}\left[P_t^{-\epsilon}\varphi_{t+k}Y_{t+k}\right]}{\mathrm{E}_t\sum_{k=0}^{\infty}\phi^k\Lambda_{t,t+k}\left[P_{t+k}^{-\epsilon}Y_{t+k}\right]}$$

$$\log(A_t) = \rho\log(A_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0,\sigma^2).$$

8.3.2 Marginal costs and output gap

Output gap

Instead *real* marginal costs, we often want express this in terms of *output gap*. To see how they are related, let's start from the definition of real marginal cost

$$MC_t = \frac{W_t/P_t}{A_t}$$

Household's first order condition with respect to labour gives

$$\frac{W_t}{P_t} = C_t^{\sigma}\theta(1-N_t)^{-\xi}$$

Subsitute this to the above marginal cost equation and use $Y_t = C_t$

$$MC_t = \frac{Y_t^{\sigma}\theta(1-N_t)^{-\xi}}{A_t}$$

The linearized version is as follows

$$\widehat{MC}_t \equiv \frac{MC_t - 1}{1} = \sigma \hat{Y}_t + \xi \frac{1}{1-N} \hat{N}_t - a_t.$$

The aggregate production function is

$$Y_t = \frac{A_t N_t}{\text{price dispersion}}$$

and linearized (the price dispersion is of second order and vanishes in the linear approximation)

$$\hat{Y}_t = a_t + \hat{N}_t$$

solve $\hat{N}_t$ from above

$$\hat{N}_t = \hat{Y}_t - a_t$$

and substituting this to above will give

$$\widehat{MC}_t = \sigma \hat{Y}_t + \xi \frac{1}{1-N} (\hat{Y}_t - a_t) - a_t.$$

and simplify to

$$\widehat{MC}_t = \left(\sigma + \xi \frac{1}{1-N} \right) \hat{Y}_t - \left(1 + \xi \frac{1}{1-N} \right) a_t. \tag{8.11}$$

The *output gap* $\tilde{Y}_t$ is the difference between the actual level of output $\hat{Y}_t$ and the "flexible price" level of output $\hat{Y}_t^f$

$$\tilde{Y}_t \equiv \hat{Y}_t - \hat{Y}_t^f.$$

Flexible price output In the flexible price economy with *perfect competition* the prices equal marginal costs, hence, real marginal cost is unity and its' deviation from steady-state (=unity) will be zero.

Then in flexible price economy the above equation writes

$$\widehat{MC}_t = 0 = \left(\sigma + \xi \frac{1}{1-N} \right) \hat{Y}_t^f - \left(1 + \xi \frac{1}{1-N} \right) a_t.$$

and, thus,

$$\hat{Y}_t^f = \frac{1 + \xi \frac{1}{1-N}}{\sigma + \xi \frac{1}{1-N}} a_t.$$

Solve for a_t and use that to eliminate a_t from the marginal cost equation (8.11)

$$\widehat{MC}_t = \left(\sigma + \xi \frac{1}{1-N}\right) \hat{Y}_t - \left(\sigma + \xi \frac{1}{1-N}\right) \hat{Y}_t^f$$
$$= \left(\sigma + \xi \frac{1}{1-N}\right) (\hat{Y}_t - \hat{Y}_t^f) \quad (8.12)$$

Denote

$$\kappa \equiv \left(\sigma + \xi \frac{1}{1-N}\right).$$

We may write the New Keynesian Phillips Curve in terms of the output gap

$$\hat{\Pi}_t = \beta \, \mathrm{E}_t \, \hat{\Pi}_{t+1} + \frac{(1-\phi)(1-\beta\phi)}{\phi} \kappa \underbrace{(\hat{Y}_t - \hat{Y}_t^f)}_{\tilde{Y}_t}.$$

Next step is to express other important equations in terms of the output gap.

Consider the log-linearized version of the consumption Euler equation (8.3)

$$\hat{C}_t = \mathrm{E}_t \, \hat{C}_{t+1} - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t \, \hat{\Pi}_{t+1} + \log \beta\right).$$

Since $Y_t = C_t$

$$\hat{Y}_t = \mathrm{E}_t \, \hat{Y}_{t+1} - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t \, \hat{\Pi}_{t+1} + \log \beta\right).$$

Add and subtract $\hat{Y}_t^f$ in the left hand side and $\mathrm{E}_t \, \hat{Y}_{t+1}^f$ in the right hand side:

$$\hat{Y}_t - \hat{Y}_t^f + \hat{Y}_t^f = \mathrm{E}_t \, \hat{Y}_{t+1} - \mathrm{E}_t \, \hat{Y}_{t+1}^f + \mathrm{E}_t \, \hat{Y}_{t+1}^f - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t \, \hat{\Pi}_{t+1} + \log \beta\right).$$

or

$$\tilde{Y}_t + \hat{Y}_t^f = \mathrm{E}_t \, \tilde{Y}_{t+1} + \mathrm{E}_t \, \hat{Y}_{t+1}^f - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t \, \hat{\Pi}_{t+1} - \bar{r}\right).$$

or when reshuffled

$$\tilde{Y}_t = \mathrm{E}_t \, \tilde{Y}_{t+1} - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t \, \hat{\Pi}_{t+1} - \bar{r} - \sigma \, \mathrm{E}_t(\hat{Y}_{t+1}^f - \hat{Y}_t^f)\right).$$

Define

$$r_t^n \equiv \bar{r} + \sigma \, \mathrm{E}_t(\hat{Y}_{t+1}^f - \hat{Y}_t^f) = \bar{r} + \sigma \frac{1 + \xi \frac{1}{1-N}}{\sigma + \xi \frac{1}{1-N}} \, \mathrm{E}_t(a_{t+1} - a_t).$$

This is called the *natural rate of interest*. It contains all the real exogenous forces in the model.

The *expectation-augmented IS curve* may be written as

$$\tilde{Y}_t = \mathrm{E}_t\,\tilde{Y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathrm{E}_t\,\hat{\Pi}_{t+1} - r_t^n\right).$$

We may now collect the log-linearized system of equations as follows

$$\hat{\Pi}_t = \beta\,\mathrm{E}_t\,\hat{\Pi}_{t+1} + \frac{(1-\phi)(1-\beta\phi)}{\phi}\kappa\tilde{Y}_t \tag{8.13}$$

$$\tilde{Y}_t = \mathrm{E}_t\,\tilde{Y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathrm{E}_t\,\hat{\Pi}_{t+1} - r_t^n\right) \tag{8.14}$$

$$r_t^n = \bar{r} + \sigma\frac{1+\xi\frac{1}{1-N}}{\sigma+\xi\frac{1}{1-N}}\,\mathrm{E}_t(a_{t+1}-a_t) \tag{8.15}$$

$$a = \rho a_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0,\sigma^2).$$

What is missing is the description how the nominal interest rate i_t is determined, ie *monetary policy*.

8.3.3 Monetary policy

Monetary policy

Determination of nominal interest rates, i_t, gives the path for actual real interest rate. It is a description how monetary policy is conducted.

Monetary policy is non-neutral

When prices are sticky, nominal interest rate path determines the output gap!

Lets denote $\pi_t \equiv \hat{\Pi}_t$ to simplify the notation.
Taylor rule

- To see how the New Keynesian economy works, let's assume the following simple interest rate rule, that is called by *Taylor rule* according to John Taylor, who prosed it in 1993

  $$i_t = \bar{r} + \phi_\pi \pi_t + \phi_y \tilde{Y}_t + v_t,$$

  where π_t is inflation and $\tilde{Y}_t$ is the output gap, and $\phi_\pi > 1$ (Taylor principle) and $\phi_y > 0$, and v_t is the monetary policy shock, that follows a stationary AR(1) process

  $$v_t = \rho_v v_{t-1} + \varepsilon_t^v, \quad \varepsilon_t^v \sim \mathrm{iid}(0,\sigma_v^2),\ |\rho_v| < 1.$$

- The first term $\bar{r}_t$ in the monetary policy rule is the steady-state real interest rate. It is determined by the rate of the time preference: $\bar{r} \equiv -\log \beta = -\log(1/(1+\rho)) \approx \rho$. This means that the nominal interest rate, in the zero inflation steady-state, will be $\bar{r}$.
- The second term is the central bank's response to the situation when inflation deviates from the target of zero (we may generalize the rule to have some other target rate of inflation, $\pi^{\star}$). The Taylor principle is important here.
- The third term captures central bank's desire to stabilize the output-gap $\tilde{Y}_t$.
- The monetary policy shock v_t can be interpreted as, potentially persistent, *surprises* that central bank creates by taking into account other factors than the current inflation and the output gap such as
  - It may have imperfect knowledge about output gap.
  - Central bank's output gap "estimate" is different from that of the other agents in the economy.
  - The monetary policy committee may disagree, and this creates surprises.
- Positive monetary policy shock, ie positive value for ε_t, would imply a *tightening* of the monetary policy. (And wise versa.)
- We have the stability requirement (similar to Taylor principle):

$$\lambda(\varphi_\pi - 1) + (1-\beta)\varphi_y > 0 \tag{8.16}$$

We solve the model shock-by-shock using the method of *undetermined coefficients*. We first study the effects of the monetary policy shock:

- Substitute the the Taylor rule to the expectation-augmented IS curve

$$\tilde{Y}_t = \mathrm{E}_t\, \tilde{Y}_{t+1} - \frac{1}{\sigma}\left(\bar{r} + \phi_\pi \pi_t + \phi_y \tilde{Y}_t + v_t - \mathrm{E}_t\, \pi_{t+1} - r_t^n\right)$$

- Let's assume that *no technology shock* occur, ie $a_t = 0$. Then $r_t^n = \bar{r}$ and above simplifies to

$$\tilde{Y}_t = \mathrm{E}_t\, \tilde{Y}_{t+1} - \frac{1}{\sigma}\left(\phi_\pi \pi_t + \phi_y \tilde{Y}_t + v_t - \mathrm{E}_t\, \pi_{t+1}\right)$$

- There is no constant term, so it is reasonable to guess that the solution takes the form
$$\tilde{Y}_t = \varphi_y v_t, \quad \pi_t = \varphi_\pi v_t.$$
Substitute this to the Phillips Curve and IS curve
$$\varphi_\pi v_t = \beta \operatorname{E}_t \varphi_\pi v_{t+1} + \frac{(1-\phi)(1-\beta\phi)}{\phi}\kappa\varphi_y v_t$$
$$\varphi_y v_t = \operatorname{E}_t \varphi_y v_{t+1} - \frac{1}{\sigma}\left(\phi_\pi\varphi_\pi v_t + \phi_y\varphi_y v_t + v_t - \operatorname{E}_t \varphi_\pi v_{t+1}\right)$$

- Note that $\operatorname{E}_t v_{t+1} = \rho_v v_t$. Use this to above to obtain
$$\varphi_\pi v_t = \beta\varphi_\pi\rho_v v_t + \frac{(1-\phi)(1-\beta\phi)}{\phi}\kappa\varphi_y v_t$$
$$\varphi_y v_t = \varphi_y\rho_v v_t - \frac{1}{\sigma}\left(\phi_\pi\varphi_\pi v_t + \phi_y\varphi_y v_t + v_t - \varphi_\pi\rho_v v_t\right)$$
Collect the terms
$$\left\{\varphi_\pi - \beta\varphi_\pi\rho_v - \frac{(1-\phi)(1-\beta\phi)}{\phi}\kappa\varphi_y\right\} v_t = 0$$
$$\left\{\varphi_y - \varphi_y\rho_v + \frac{1}{\sigma}\left(\phi_\pi\varphi_\pi + \phi_y\varphi_y + 1 - \varphi_\pi\rho_v\right)\right\} v_t = 0$$

- The next step is to solve φ_π and φ_y from the following system of equations
$$\varphi_\pi - \beta\varphi_\pi\rho_v - \frac{(1-\phi)(1-\beta\phi)}{\phi}\kappa\varphi_y = 0$$
$$\varphi_y - \varphi_y\rho_v + \frac{1}{\sigma}\left(\phi_\pi\varphi_\pi + \phi_y\varphi_y + 1 - \varphi_\pi\rho_v\right) = 0.$$
Collect the terms:
$$\varphi_\pi\left[1-\beta\rho_v\right] - \underbrace{\frac{(1-\phi)(1-\beta\phi)}{\phi}\kappa}_{\equiv\lambda}\varphi_y = 0$$
$$\varphi_y\left[1-\rho_v+\frac{1}{\sigma}\phi_y\right] + \varphi_\pi\left[\frac{1}{\sigma}(\phi_\pi-\rho_v)\right] + \frac{1}{\sigma} = 0.$$
The first equation results
$$\varphi_\pi = \frac{\lambda}{1-\beta\rho_v}\varphi_y.$$

Substitute this into the second equation

$$\varphi_y \left[1 - \rho_v + \frac{1}{\sigma} \phi_y \right] + \frac{\lambda}{1 - \beta \rho_v} \varphi_y \left[\frac{1}{\sigma} (\phi_\pi - \rho_v) \right] + \frac{1}{\sigma} = 0$$

Multiply both sides by σ

$$\varphi_y \left[\sigma(1 - \rho_v) + \phi_y \right] + \frac{\lambda}{1 - \beta \rho_v} \varphi_y \left[(\phi_\pi - \rho_v) \right] + 1 = 0$$

and by $1 - \beta \rho_v$

$$\varphi_y (1 - \beta \rho_v) \left[\sigma(1 - \rho_v) + \phi_y \right] + \lambda \varphi_y \left[(\phi_\pi - \rho_v) \right] + 1 - \beta \rho_v = 0$$

Collect terms

$$\varphi_y \underbrace{\left\{ (1 - \beta \rho_v) \left[\sigma(1 - \rho_v) + \phi_y \right] + \lambda \left[(\phi_\pi - \rho_v) \right] \right\}}_{\equiv \Lambda_v} + 1 - \beta \rho_v = 0$$

and, finally, solve φ_y

$$\varphi_y = -\frac{(1 - \beta \rho_v)}{\Lambda_v}$$

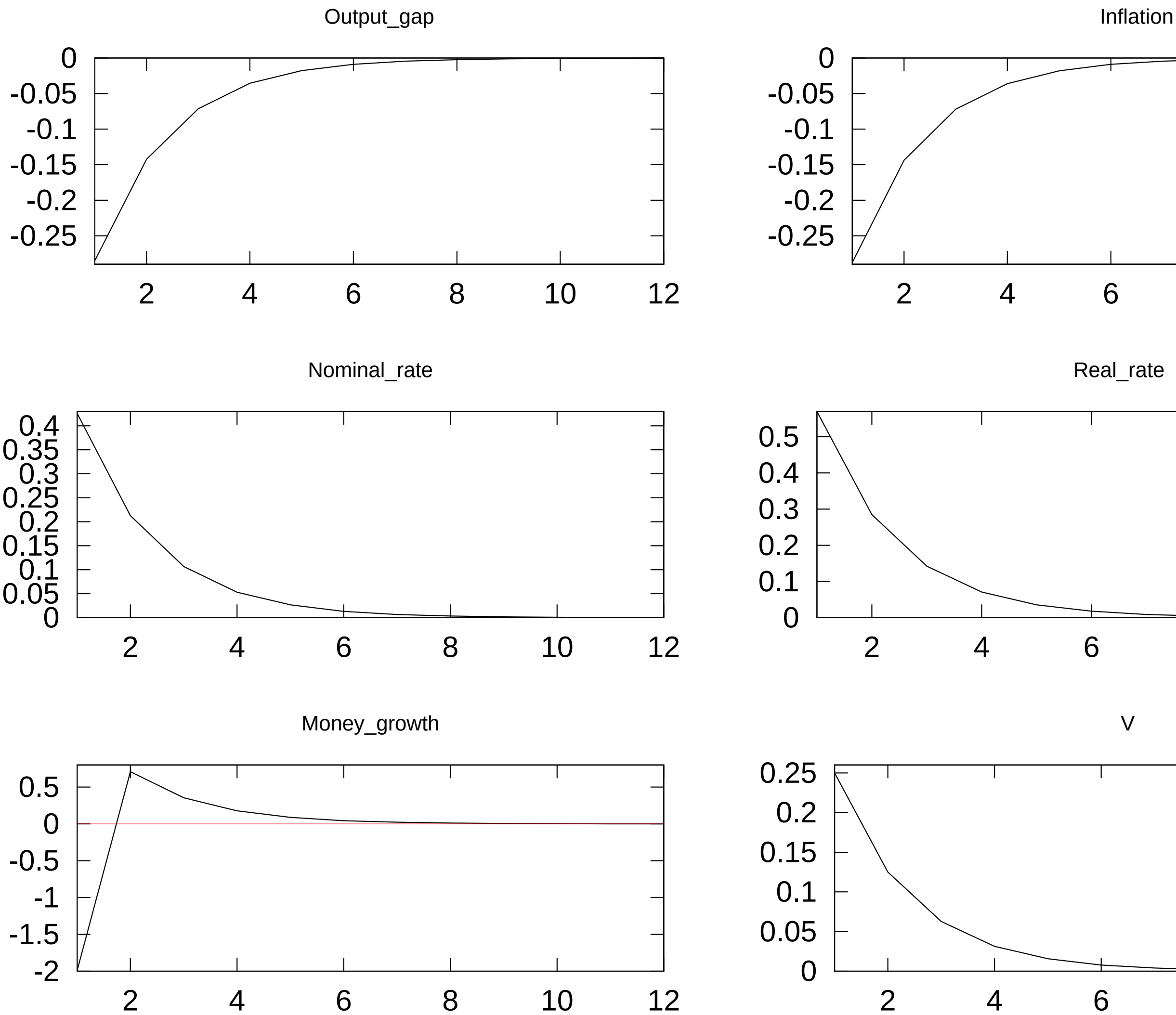
and, from the first equation, φ_π

$$\varphi_\pi = -\frac{\lambda}{\Lambda_v}.$$

Hence, the *solution* is

$$\pi_t = -\frac{\lambda}{\Lambda_v} v_t$$
$$\tilde{Y}_t = -\frac{(1 - \beta \rho_v)}{\Lambda_v} v_t.$$

If (8.16) holds, $\Lambda_v > 0$.



Discussion:

- We may, naturally, use the solution to calculate the impulse responses.
- Knowing this solution, we may calculate the solution of other variables like i_t, r_t or $\hat{Y}_t$ (note that at we assume $a_t = 0$ $(\forall t)$ and therefore $\hat{Y}_t^f$ is zero).
- If there is a positive monetary shock that results positive values of v_t, the inflation will decline both in inflation and in output gap.
- Decline in output gap $\hat{Y}_t - \hat{Y}_t^f$ means also that actual output will decline.

$\longrightarrow$Monetary policy is *non-neutral*, ie monetary policy will affect the real variables, ie Phillips-curve is upward-sloping!

- Since v_t's response to the shock innovation ε_t is persistent, ie it returns back to (zero) steady-state gradually, the response of inflation and output gap will also be persistent. So, they inherit the persistence of v_t.

- The response of real interest rate may be calculated from (8.14)

$$r_t - \bar{r} = \frac{\sigma(1-\rho_v)(1-\beta\rho_v)}{\Lambda_v} v_t$$

It is going to increase as a result of a positive monetary policy shock.

- If ρ_v is very high, the positive monetary shock may result a decline in i_t. To see this, consider the Fisher equation, and substitute the solutions of π_t and r_t as follows

$$i_t = r_t + \mathrm{E}_t\,\pi_{t+1} = \frac{\sigma(1-\rho_v)(1-\beta\rho_v) - \rho_v\lambda}{\Lambda_v} v_t$$

The intuition is the following: if monetary shock is very persistent, ie high ρ_v, the agents know that the real interest rate will stay high for a long period. Firms that have given the Calvo signal — who can change their price — respond to this by adjusting the price level downwards. Those who cannot adjust has too high price (relative to the rest of the economy) and face demand decline. Due to high real interest rates, households save more and, therefore, postpone their consumption.

Impact of the technology shock

To study the technology shock analytically, we shut down the monetary policy shock by assuming $v_t = 0$.

Remember that the technology process is defined as

$$a = \rho a_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0, \sigma^2)$$

Remember also that the real natural interest rate is given by

$$r_t^n = \bar{r} + \sigma \frac{1 + \xi\frac{1}{1-N}}{\sigma + \xi\frac{1}{1-N}} \mathrm{E}_t(a_{t+1} - a_t) = \bar{r} + \sigma\chi\,\mathrm{E}_t(a_{t+1} - a_t),$$

where

$$\chi \equiv \frac{1+\xi\frac{1}{1-N}}{\sigma+\xi\frac{1}{1-N}}$$

is positive. Utilizing $\mathrm{E}_t\, a_{t+1} = \rho a_t$ we get

$$r_t^n - \bar{r} = -\sigma\chi(1-\rho)a_t.$$

We solve the system using, again, the method of undetermined coefficients (or using utilizing above solution while $r_t^n - \bar{r}$ enters to the system in a similar manner as v_t above but with the opposite sign.

This will result the following solution

$$\begin{aligned}
\tilde{Y}_t &= (1-\beta\rho)\Lambda_a(r_t^n - \bar{r}) \\
&= -\sigma\chi(1-\rho)(1-\beta\rho)\Lambda_a a_t \\
\pi_t &= \lambda\Lambda_a(r_t^n - \bar{r}) \\
&= -\sigma\chi(1-\rho)\lambda\Lambda_a a_t,
\end{aligned}$$

where

$$\Lambda_a = \left[(1-\beta\rho)\left[\sigma(1-\rho)+\phi_y\right] + \lambda(\phi_\pi - \rho)\right]^{-1}.$$

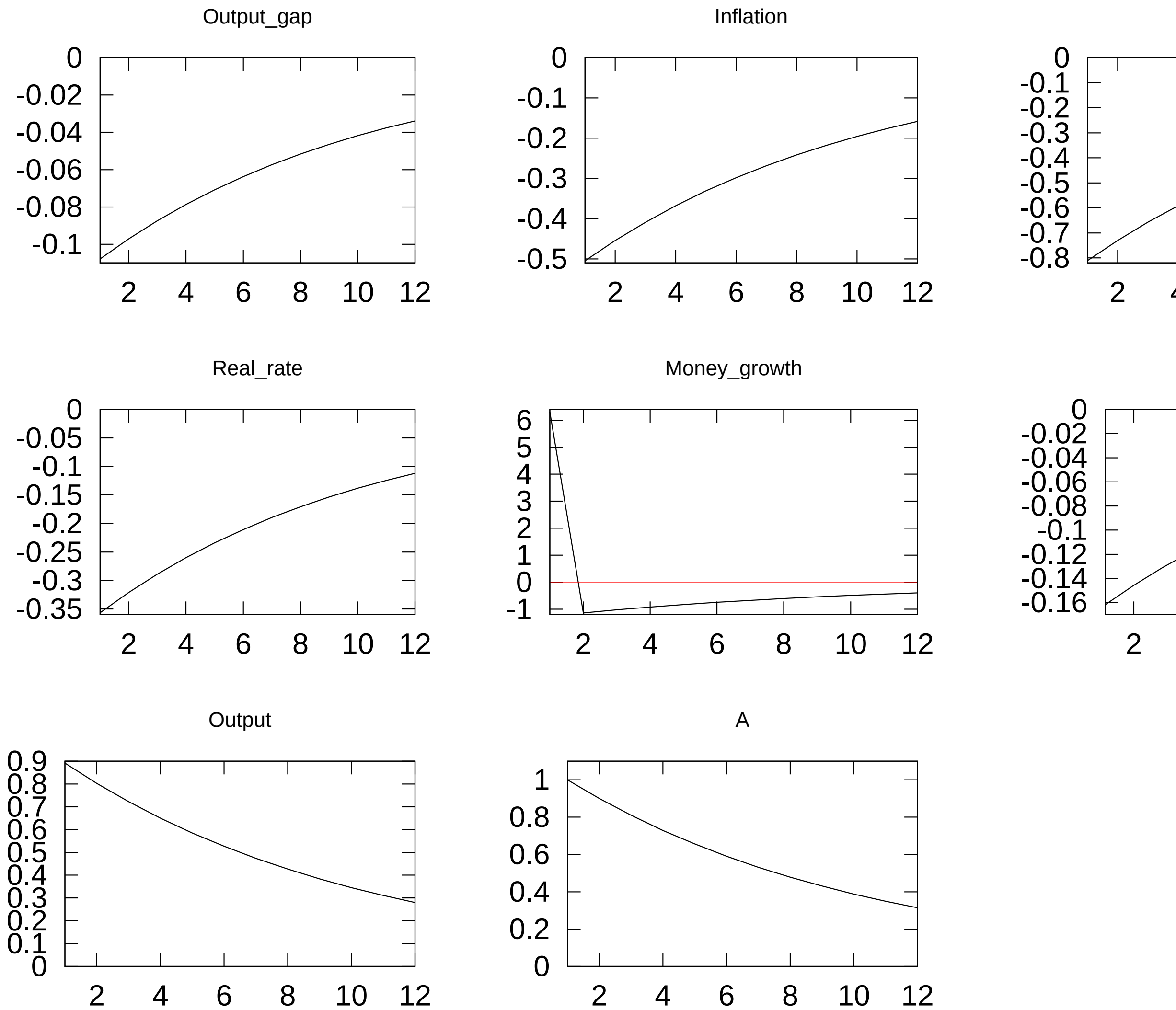
Again, given the stability condition (Taylor principle), $\Lambda_a > 0$.

We may solve the output from the definition of the output gap

$$\hat{Y}_t = Y_t^f + \tilde{Y}_t = \chi[1-\sigma(1-\rho)(1-\beta\rho)\Lambda_a]a_t$$

and employment from the aggregate production function

$$\hat{N}_t = \hat{Y}_t - a_t = [(\chi-1) - \sigma\chi(1-\rho)(1-\beta\rho)\Lambda_a]\, a_t$$



Discussion (of the positive technology shock)

- Response of inflation is negative. Technological improvement drives down the marginal costs and, sluggishly, the optimal price leading to lower inflation.
- However, the response of output *gap* is *negative*. This is due to the fact that the flexible price output increases much faster (following the technology shock) than the actual output! The response of actual output is sluggish since the firms may adjust their prices (downwards) in a slower pace due to the price rigidities.
- The response of output is ambiguous depending on the parameter

values, in particular the parameters of the Taylor (monetary policy) rule. If $\sigma = 1$ the response will be positive.

- The real *natural* interest rate response is negative since the central bank responds to inflation and output gap decline by lowering the nominal interest rate *more than the decline in inflation*!

- Following the same logic than the case of the output response, the employment/hours response is also ambiguous. It would be negative if $\sigma = 1$.

8.3.4 Optimal monetary policy

Distortions
Two deviations from efficient competitive economy

Imperfect competition
Price level is higher than in a competitive economy. Real wages and employment are lower.

Price rigidities
Firms that cannot change their price have their prices in suboptimal level
$\longrightarrow$make losses when the shock hits.

Relative price distortions: There can exist firms that have not allowed to change their prices a very long time. In general, the relative prices of individual goods deviate each other (despite of similar technologies).

Solutions to distortions
The imperfect competition could be corrected by an employment subsidy.
$\longrightarrow$Brings the economy's output, wages and employment to the level of competitive economy.
Optimal policy would set inflation to the level where no firm that is in need to change their prices when allowed to change it.
$\longrightarrow$Set inflation to zero, ie keep price level unchanged.
$\longrightarrow$Output would reach the efficient (non-price-rigidity) level.
$\longrightarrow$Output gap would be zero!
Discussion

- We assumed that employment subsidy kills the distortions resulted from imperfect competition.

- By setting inflation to zero, efficient level of output would be reached.

- All prices at optimal level
  ⟶No variations in the markup.
- Stabilization of the output is not desirable *per se*. It would follow the efficient level.
- Aggregate price level is not a policy *target*!
- It arises in making all firms content with their existing prices.
- To set interest rates optimally requires knowledge of the natural real rate of interest r_t^n.
- It contains unobserved shocks and requires knowledge of the model and its parameters
  ⟶Hard to know in real economies
  ⟶Search for simple, robust rules that generate reasonable outcome in wide range of models
  ⟶Maybe simple Taylor rule is a one! (Not everybody believes this.)

9 Monetary policy at the zero lower bound of nominal interest rates

Inflation dynamics

Consider the consumption Euler equation that was studied in the previous section:

$$\frac{1}{1+i_t} = \beta\, \mathrm{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right\}$$

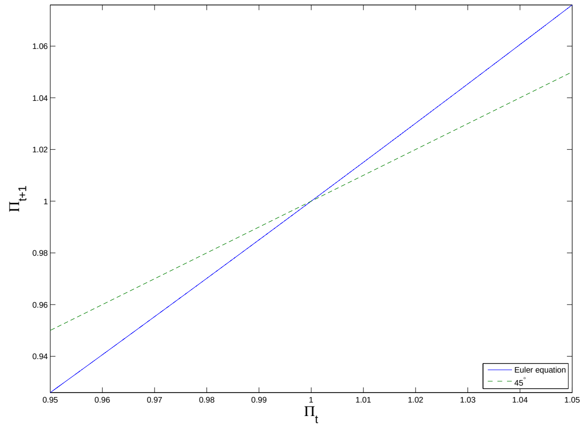
and the simple Taylor -type rule (without output gap) in the *non*-linearized form

$$1 + i_t = \beta^{-1} \Pi_t^{\phi_\pi}.$$

Consider the deterministic case (drop expectation operator) and substitute interest rate out from consumption Euler equation using the Taylor rule to obtain

$$\Pi_t^{\phi_\pi} = \left(\frac{C_{t+1}}{C_t} \right)^{\sigma} \Pi_{t+1}$$

Phase diagram



$\sigma = 2$, $C_{t+1}/C_t = 1$, $\phi_\pi = 1.5$. The steady state is $\Pi = 1$

Inflation dynamics and the zero lower bound (NLB)

Consider the case where the nominal interest rate is bounded by zero. This is due to existence of paper money (that we do not model here).

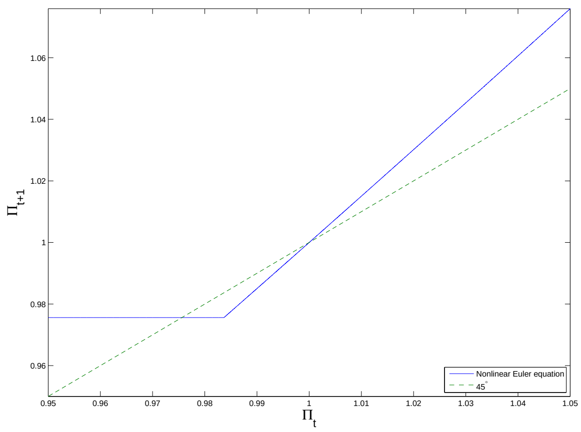
The Taylor rule can be modified according to

$$1 + i_t = \max\left(1, \beta^{-1}\Pi_t^{\phi_\pi}\right).$$

and the resulting Euler equation

$$\max\left(1, \beta^{-1}\Pi_t^{\phi_\pi}\right) = \frac{1}{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{\sigma}\Pi_{t+1}$$

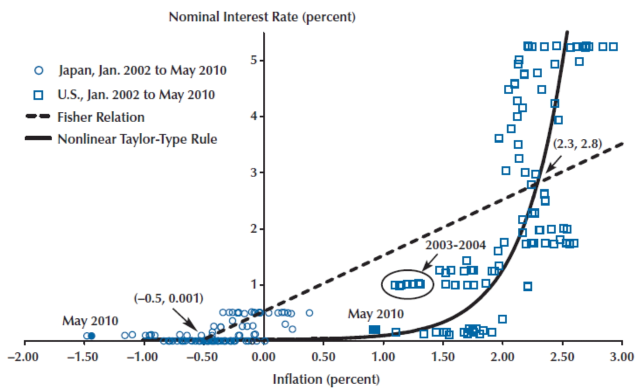
Phase diagram with two steady states



In the lower steady state $i = 0$: $\Pi = \beta \left(C_{t+1} / C_t \right)^{-\sigma} = \beta$, resulting $\pi = -\rho$.

Interest rates and inflation in Japan and US

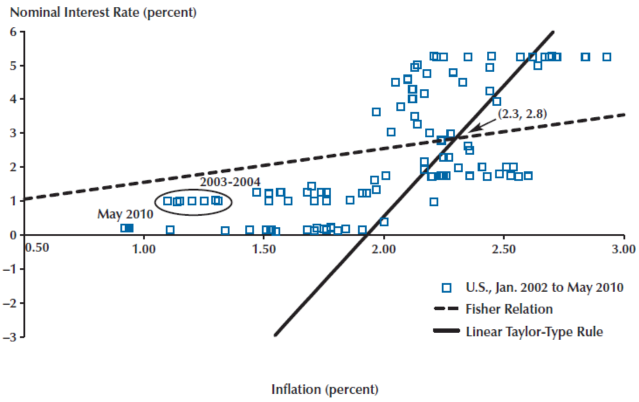
Interest Rates and Inflation in Japan and the U.S.



NOTE: Short-term nominal interest rates and core inflation rates in Japan and the United States, 2002-10.
SOURCE: Data from the Organisation for Economic Co-operation and Development.

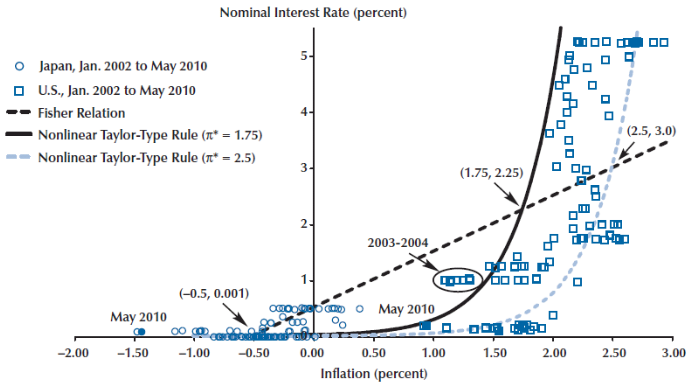
Source: Jim Bullard (2010)

Denial



Increasing inflation target

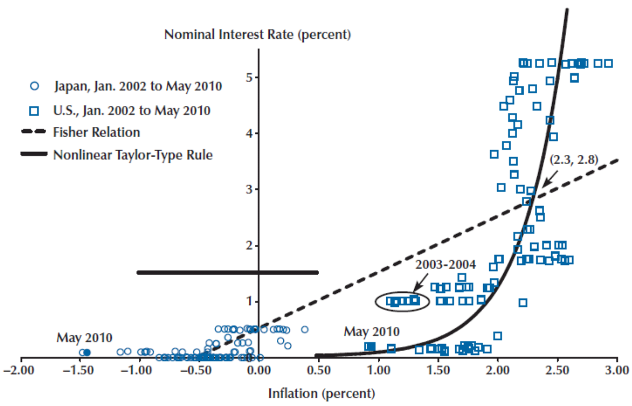
Inflation Expectations, Interrupted



NOTE: The 2003-04 episode. Thornton (2006, 2007) argues that FOMC communications increased the perceived inflation target of the Committee.

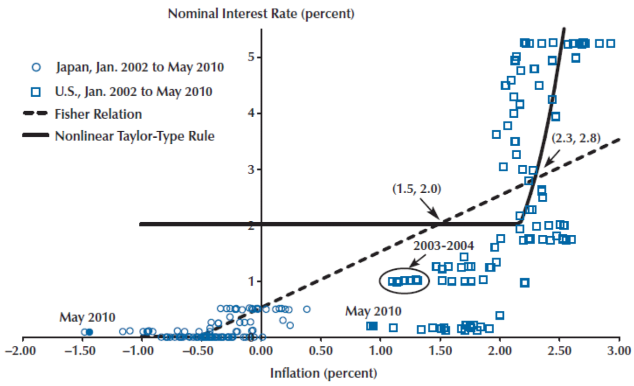
Discontinuity

Discontinuity



NOTE: The discontinuous Taylor-type policy rule looks unusual but eliminates the unintended steady state.

Historic policy



Other proposals

Government insolvency

- Government can threaten to behave in an insolvent manner in the case of lower steady-state.
- "Unsophisticated implementation".
- Japanese aggressive fiscal policy (debt-to-GDP ratio over 200 %)

Quantitative easing

- Long-term rates falling: US, UK, euro area
- Permanent or temporary asset purchases

- General problem: what is inflation expectations will anchor to the lower steady-state.

THE END

Thank you for your attention.
Please report the errors, typos, etc to `antti@ripatti.net`

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