One-dimensional fragment of first-order logic

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Recent work on two-variable logic

- Benaim et al. Complexity of two-variable logic on finite trees. LICS 2013.
- Charatonic and Witkowski. Two-variable logic with counting and trees. LICS 2013.
- Swast and Tendera. Two-variable logic with one transitive relation is decidable. STACS 2013.
- Kieroński et al. Two-variable first-order logic with equivalence closure. LICS 2012.
- etc. etc. etc.

Uniform one-dimensional fragment

- $\mathrm{UF}_1^=$ canonically generalizes two-variable logic.
- Allows meaningful use of atoms of higher arities.
- $\mathrm{NEXPTIME}$-complete SAT and FINSAT.
- Incomparable with guarded negation fragment and two-variable logic with counting.

Uniform one-dimensional fragment

- Restricts quantification to blocks $\overline{\exists x}$ and $\overline{\forall x}$ that leave at most one variable free.
- An additional uniformity condition holds: Boolean combinations of atoms $Rx_1, \ldots, x_k$ and $Sy_1, \ldots, y_l$ of higher arities is allowed only if $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_l\}$.
- Closed under Boolean operators and free use of identity.

Let X be a set of variables. An X-atom is an atomic formula whose set of variables is exactly X.

Rxy and $Txyyxx$ are $\{x, y\}$-atoms. $Sxyz$ and Uxx are not.

Syntax

1. $(P \in \tau$ and $x \in \mathrm{VAR}) \;\Rightarrow\; Px \;\in\; \mathrm{UF}_1^=$
2. $(R \in \tau$ and $x \in \mathrm{VAR}) \;\Rightarrow\; Rxxxxx \;\in\; \mathrm{UF}_1^=$
3. $x, y \in \mathrm{VAR} \;\Rightarrow\; x = y \in \mathrm{UF}_1^=$
4. $\varphi \in \mathrm{UF}_1^= \;\Rightarrow\; \neg\varphi \in \mathrm{UF}_1^=$
5. $\varphi, \psi \in \mathrm{UF}_1^= \;\Rightarrow\; (\varphi \wedge \psi) \in \mathrm{UF}_1^=$
6. Let $X := \{x_0, ..., x_k\} \subseteq \mathrm{VAR}$. Let φ be a Boolean combination X-atoms and formulae of $\mathrm{UF}_1^=$ with the free variables in X. Then $\exists x_1 ... \exists x_k\, \varphi \;\in\; \mathrm{UF}_1^=$.
7. Also $\exists x_0 ... \exists x_k\, \varphi \;\in\; \mathrm{UF}_1^=$.

OK:

$$\exists y \exists z \Big((\neg Rxyz \vee Tyyxxz) \wedge (Px \rightarrow Qy) \Big).$$

$$\exists x \exists y \exists z \Big((\neg Rxyz \vee \neg x = y) \rightarrow (Syzx \rightarrow Qy) \Big).$$

$$\exists x \exists y \exists z \Big((\neg Rxyz \vee \neg x = y \vee \varphi(y)\,) \rightarrow Syzx \Big).$$

Not OK:

$$\exists y \exists z \Big((\neg Rxyz \vee Uxz) \wedge (Px \rightarrow Qy) \Big).$$

$$\exists z \Big((\neg Rxyz \vee \neg x = y) \rightarrow (Syzx \rightarrow Qy) \Big).$$

Proposition

$\mathrm{UF}_1^{=} \not\leq \mathrm{FOC}^2$ *and* $\mathrm{UF}_1^{=} \not\leq \mathrm{GNF}$

Proof.

$\exists x \exists y \exists z \neg Rxyz$ is in $\mathrm{UF}_1^{=}$. □

Proposition

$\mathrm{FOC}^2 \not\leq \mathrm{UF}_1^{=}$ *and* $\mathrm{GNF} \not\leq \mathrm{UF}_1^{=}$

Proof.

Harder. □

$\mathrm{UF}_1^=$ is based on uniformity and one-dimensionality. How about loosening these conditions?

General one-dimensional fragment

1. $(P \in \tau$ and $x \in \mathrm{VAR}) \ \Rightarrow\ Px \ \in\ \mathrm{GUF}_1$
2. $(R \in \tau$ and $x \in \mathrm{VAR}) \ \Rightarrow\ Rxxxxx \ \in\ \mathrm{GUF}_1$
3. $\varphi \in \mathrm{UF}_1^{=} \ \Rightarrow\ \neg\varphi \in \mathrm{GUF}_1$
4. $\varphi, \psi \in \mathrm{UF}_1^{=} \ \Rightarrow\ (\varphi \wedge \psi) \in \mathrm{GUF}_1$
5. Let φ be a Boolean combination of atoms and formulae of GUF_1 with the free variables in X. Then $\exists x_1 ... \exists x_k\, \varphi \ \in\ \mathrm{GUF}_1$.
6. Also $\exists x_0 ... \exists x_k\, \varphi \ \in\ \mathrm{GUF}_1$.

Strongly uniform two-dimensional fragment

Strongly uniform two-dimensional fragment SUF_1 is defined as $\mathrm{UF}_1^=$, but such that a block of quantifiers can leave up to two variables free. Also further restrictions on the syntax are imposed.

Proposition

General one-dimensional fragment is undecidable.

Theorem

Strongly uniform two-dimensional fragment is undecidable.

$\mathrm{UF}_1^=$ is NEXPTIME-complete and contains FO^2. Also FOC^2 is NEXPTIME-complete. How about adding counting quantifiers $\exists^{\geq k}x$ to $\mathrm{UF}_1^=$?

Theorem (K. and Kieroński, 2014)

UFC_1 *is undecidable. (SAT is* Π_1^0*-complete and FINSAT* Σ_1^0*-complete.)*

$\mathrm{UF}_1^=$ has a canonical modal normal form. Polyadic diamonds can be formed by allowing the Boolean combination, permutation, and variable identification of relations.

1. Let $X = \{x_1, ..., x_k\}$ be a set of variables. Then any finite Boolean combination of X-atoms and identities in the variables $x_1, ..., x_k$ defines a k-ary relation $\mathcal{R}$. (This requires an ordering of the variables.)
2. Any such relation $\mathcal{R}$ is a diamond.

$$\varphi \ ::= \ P \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\mathcal{R}\rangle(\varphi_1, ..., \varphi_{k-1}),$$

where P is a unary relation and $\mathcal{R}$ defines a diamond.

Proposition

Let φ be a formula of $\mathrm{UF}_1^=$ with exactly one free variable. Then φ admits a representation in the above modal normal form.

Generalizes a result from C. Lutz, U. Sattler, and F. Wolter. *Modal logic and the two-variable fragment.* CSL 2001.

Theorem

$\mathrm{UF}_1^=$ *is* $\mathrm{NEXPTIME}$*-complete.*

Proof.

The most interesting proof concerns the equality-free fragment of $\mathrm{UF}_1^=$ and a number of extensions of it.

The proof is based on a combination of Morleysation and a novel torus filtration technique that allows one to deal with logics with diamonds $\langle R \rangle$ and $\langle \overline{R} \rangle$.

Torus filtration is a generalization of the standard filtration based on using a finite number of copies of the original domain.

This technique is ultimately based on a reduction to the monadic class of first-order logic. □