On intractable tracks

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Abstract: The principle of least action in its original form á la Maupertuis can be understood as the universal quest to consume free energy in the least time. The ensuing dispersion of energy is a nondeterministic process when there are various ways to consume free energy because the flows of energy and their driving forces cannot be separated so as to track the motions by way of integration. Despite this resulting intractability, the natural processes are not random, but the flows of energy themselves will naturally select the least-action paths from available variations. In this way, high-density closed actions will process step by step to lower and lower densities by opening up and expelling one or multiple elements of the absolutely least action, scaling to Planck's constant. Ultimately the spontaneous symmetry-breaking process will terminate, when the most elementary action in a closed form opens up and transforms to the action of the lowest symmetry, the photon. Consequently, when all entities in nature are described as actions, there is no fundamental difference between particles and their associated forces. The holistic worldview provided by the principle of least action in its mathematical form allows us to understand why nature displays rules and regularities but is nevertheless unpredictable. © *2012 Physics Essays Publication*. [DOI: 10.4006/0836-1398-25.2.233]

Résumé: Le principe de moindre action dans sa forme originale, chez Maupertuis, peut être compris comme la quête universelle de la consommation de l'énergie libre dans le temps le plus bref. La dispersion d'énergie qui s'ensuit est non-déterministe quand il y a des chemins variés pour la consommation de l'énergie libre parce qu'alors les flux d'énergie et les forces motrices ne peuvent pas être séparées pour tracer les mouvements en utilisant l'intégration. En dépit de cette indocilité résultant, les flux d'énergie eux-mêmes sélectionneront les chemins de moindre action au milieu des possibilités disponibles. De cette façon, les actions des particules de haute densité progresseront pas à pas vers des densités de plus en plus faibles en s'ouvrant et en expulsant un ou plusieurs éléments de la moindre action absolue, qui est étalonnée sur la constante de Planck. Ultimement ce processus de rupture spontanée de la symétrie se terminera lorsque l'action la plus élémentaire dans une forme de particule s'ouvre et se transforme dans l'action de symétrie la plus faible, le photon. Conséquemment, quand toutes les entité dans la nature sont décrites comme actions, il n'y a pas de différence fondamentale entre particules et leurs forces associées. La vision du monde holistique fournie par le principe de moindre action dans sa forme mathématique nous permet de comprendre pourquoi la nature montre des règles et des régularités mais reste cependant imprévisible.

Key words: Energy Dispersal; Evolution; Free Energy; Hierarchy; Principle of Least Action; Quantization.

I. INTRODUCTION

The rules and regularities we see in all scales of nature suggests that there exists an underlying organizing principle. The scale-free dependencies, most notably power laws, as well as skewed distributions of animate and inanimate species are ubiquitous. In this respect the principle of least action was early on thought of as a powerful way to make sense out of complex phenomena as well as of simple matters.[1] However, its mathematical form, as we know it today,[2,3] reflects more our desire to predict prospects than it does our urge to understand nature. This distinction may at first seem bizarre because we might reasonably suppose that complete comprehension would afford precise predictions. But the distinction between comprehension and predictability is already apparent from the course of the history that shaped the variations principle.

The following brief historical account will clarify that Maupertuis's formulation of the principle of least action, albeit intractable, is an accurate description of natural processes, whereas the modern Hamiltonian form, though tractable, is limited to describing constant-energy dynamics. This distinction was recognized already early on,[4] but its implications seem to go unrecognized today.

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II. FROM EVOLUTIONARY OPENINGS TO CONSERVATIVE CONFINEMENTS

Maupertuis stated that the action that brings about all changes in nature would always be the minimum possible,[1] and Leibniz reasoned that among all conceivable worlds the most favorable is what becomes actualized. These general statements are today seen as merely philosophical and were regarded already by their contemporaries as vague because the conditions to be satisfied for a solution were not forthcoming. Since particular circumstances, that is, constraints or boundary conditions, were not defined, there was no way to calculate the minimum action among alternatives. Therefore the original form of the principle of least action was, and is still today largely, regarded as incorrect.

Maupertuis gave the action

$$S=\int \mathbf{p}\cdot d\mathbf{x}=\int \mathbf{p}\cdot\mathbf{v}dt=\int 2Kdt=\int(-\mathbf{x}\cdot\nabla U+t\partial_t Q)dt \tag{1}$$

as momentum, $\mathbf{p}=m\mathbf{v}$, over a path $d\mathbf{x}$ to be minimized.[5] Euler made an equivalent statement of the variational principle[6] as an integral over time dt, where velocity $\mathbf{v}$ $(=d_t\mathbf{x})$ is contained in *vis viva*, $2K(t)=mv^2$. As is well known, König[7] pointed out to Maupertuis that Leibniz had already earlier given the same form to be either maximized or minimized.[8] Today we recognize that Leibniz's *vis viva* is twice the kinetic energy K. The objective of any variational principle is to find the path where the integral is an extremum. However, Maupertuis did not require that the action would be stationary, which is opposite to the stance of Lagrange and Hamilton. In other words, the Maupertuis form allows the kinetic energy $2K(t)$ to change due to energy influx or efflux when the system moves from one state to another. Due to the net flux of energy from the surroundings to the system or vice versa, the boundary conditions keep changing, affecting the motion itself. Hence the motion will be path dependent, and the end point of motion cannot be defined beforehand. Since the motion is consuming its driving forces, the motion and its driving forces depend on each other, and these variables cannot be separated to predict the trajectory by calculation. This inability to predict a future path is referred to here as intractability.

Apart from being noncomputable the Maupertuis action is a mathematically sound formulation of open systems. Most importantly, it complies with the conservation of energy, including *both* the system and it surroundings, so that the change in kinetic energy $2K$ is balanced by changes in the scalar U and vector potential Q as has been conjectured.[9–10] In contrast when the conservation of energy is required *only* within the system, the modern principle of stationary action will emerge. Since the Lagrange-Hamilton formalism is devoid of net dissipation, it cannot describe a change of state, but it can be used to compute the stationary-state trajectory by finding the minimum of the integral among conceivable paths.

Euler struggled with the intractable character of Maupertuis action. He was able to identify a special circumstance where the generally noncomputable action is computable,[4] but Lagrange first succeeded in expressing a definite formula using his newly developed calculus of variations. This Lagrangian form was achieved by requiring a certain condition, namely, that the energy of the system be defined. Then it was possible to deduce mathematically among all those paths that may bring a system from an initial position to a given final position the trajectory corresponding to minimum action.[2,6] In other words, Lagrange's formalism does not describe the actual physical system when searching out its optimal trajectory, but the calculus of variations is used only as a mathematical method to find the minimum action. Finally Hamilton revised the Lagrangian to the modern form of the least-action principle where the definite time integral is taken over the difference between the kinetic K and potential energy U:

$$S=\int_{t_1}^{t_2} Ldt=\int_{t_1}^{t_2}(K-U)dt, \tag{2}$$

whose minimum is found from the condition $\partial_x L-d_t\partial_v L=0$.[3] Hamilton's form does not contain a vector potential that would account for net influx or efflux between the system and its surroundings, such as light. In those cases when the vector potential is included in the momentum, such as was done by Dirac, the Hamiltonian is still kept conserved, and that formalism thereby limits itself to determining the stationary-state modes of motion. Therefore the modern form of the principle of least action is a consistent and computable account of stationary-state systems, but it is largely incompetent in rationalizing the behavior of dissipative systems, for example, biological systems.[11]

This brief account of the history of variations principle outlines how the principle of least action became transformed from its general form (Eq. 1), which includes open and path-dependent natural processes, to its present specific formula (Eq. 2), which is limited to stationary-state dynamics along bounded and hence determined trajectories (Fig. 1). Apparently it was merely the desire to calculate trajectories rather than the rationale to accept them as intractable that narrowed the general principle to a specific law. Thus, there is mathematically nothing wrong in the Maupertuis form. Indeed, intractability is recognized as a profound property of a mathematical form and is today acknowledged as a key problem in mathematical physics and the information sciences, such as in the form of the Navier-Stokes equation and the P versus NP problem.

III. DETERMINED DESTINATIONS AND OPEN OPTIONS

The mentioned assertion that the modern Lagrange-Hamilton form (Eq. 2) restricts the principle of least action to cases where, in fact, there is nothing that needs to be predicted may at first seem odd, but then obvious.[12]

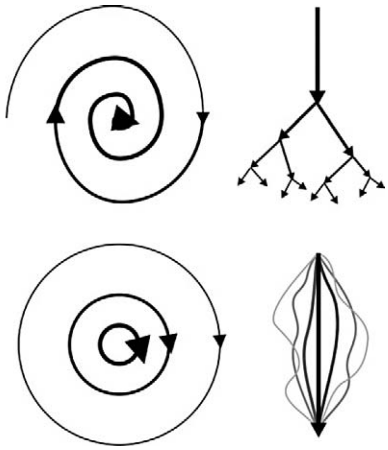

FIG. 1. The open actions mathematically formulated by Maupertuis describe natural processes that spiral irreversibly or proceed nondeterministically along branching tracks of, e.g., phylogenesis, when consuming free energy in least time (above). In contrast the bound actions formulated by Hamilton describe deterministic processes that either orbit reversibly along stationary trajectories or consume free energy along trajectories without alternative destinations (below).

When a given path is bound, then, by definition, the destination is decided beforehand, and when a path is closed, then the return is certain. Technically speaking, the paths of holonomic systems are integrable and continuously differentiable, that is, path independent. This condition of computability is proven by the Taniyama-Shimura conjecture, which states that for every elliptic curve there is a modular form of a Dirichlet L-series.[13] When the amount of energy that will be either given to, or taken from, a system, is defined, then the final state is defined relative to the initial state. In this special case, the least-action transition trajectory can be calculated. Also, when energy is fixed within the system, the conserved system cannot grow or shrink over the period of integration. Since energy is constant, there is a norm, and it is possible to find a unitary transformation to a frame where isergonic motions are independent of time. This holonomic system executes reversible Lagrangian dynamics and is, in fact, in a stationary state whose eigenvalues and eigenmodes can be determined.

In contrast, when evolutionary paths are open, their destinations are undecided. At each state along an open path, a particular choice between available alternatives depends on those steps that have already been taken, as well as upon currently encountered contingencies. Technically speaking the paths of nonholonomic systems are nonintegrable, and derivatives at the branching points are inexact. This is also familiar from the notion of classical thermodynamics where the inexact differential of energy flux $đQ = TdS$ drives the change in entropy dS of a system whose average energy is denoted by temperature T. When the net energy influx to the system or efflux to the surroundings is undetermined, there is no conserved quantity, and hence no norm either that would be required for a unitary transformation to remove the time dependence.[14–17] Obviously eigenvalues and eigenmodes cannot be determined since they are changing. Therefore the nonconserved system is genuinely moving in the course of time; that is, its evolution is irreversible. There is no coordinate transformation that could remove the time dependence. Therefore it seems logical to us that the flow of time experienced by such a system follows from the flow of energy to the surroundings or vice versa.[15]

Future paths cannot be predicted when there are two or more alternative ways to consume driving forces, that is, the energy density differences between the system and its surroundings. When a motion as a flow of energy takes a path, other flows that consume the same source of free energy are affected and vice versa. This kind of intractability is familiar from the three-body problem, the protein-folding problem, the traveling salesman problem, and other computational problems.[18–22] Since an evolutionary system keeps changing by acquiring or expelling energy, its surrounding conditions will change as well, and obviously these constraints of integration cannot be defined so as to complete the path integrals. Thus it is easy to comprehend why natural processes described by the Maupertuis action are inherently intractable.

IV. NATURAL SELECTION FOR LEAST-TIME FREE-ENERGY CONSUMPTION

The mentioned classification of systems as tractable or intractable is clarifying but is not explicit in explaining from whence the rules and regularities we see in nature might follow. Although the general form of the principle of least action (Eq. 1) cannot be integrated to a closed form, we will urge that the imperative to minimize the action remains a desideratum for evolutionary processes. Nature does not “know” where it is on its way to, and motions will distribute in variable ways[23]—but in such a way that energy density differences will be consumed in the least time. For example, a stream will by the mere act of flowing search and naturally select among currently available alternatives the most optimal path, and it will moreover shape its path as well. In other words, the constraints keep changing since it is the flows of energy themselves that will mold the energy landscape. This non-Abelian character appears also as nonvanishing Lie's derivative.[24] It means that the change $\mathbf{v} = d_t\mathbf{x}$ in the coordinate and the change $\mathbf{F} = d_t\mathbf{p}$ in the momentum are not collinear due to the net energy flux $\partial_t Q$ over dt to the system from the surroundings or vice versa, whereas for tractable Euclidean systems the force $\mathbf{F} = m\mathbf{a}$ is collinear with acceleration $\mathbf{a}$.

The notion of a natural selection for the fittest (generally, the best adapted) from the available variation is at the heart of biological evolutionary theory.[25] The Maupertuis principle of least action reveals that evolution is a physical process. The species are regarded as specialized mechanisms of energy transduction. Less effective means of energy dispersal will acquire less flow or eventually be extinguished altogether. The vital quest for free energy was understood by Boltzmann, who wrote that “Available energy is the main object at stake in the struggle for existence and the evolution of the world."[26] Despite the correct comprehension about the universal objective to consume free energy, Boltzmann was unable to formulate the statistical mechanics of open systems. Also, Gibbs realized that everything that exists, animate just as inanimate, can be formulated in universal terms of energy,[27] but he too refrained from expanding the formulation for open systems to include net fluxes between the system and its surroundings. These limitations to understanding the inherent intractability of natural processes were recently clarified when analyzing the equation of motion for an open system.[15–18]

The universal law of nature to consume free energy is not only expressed by the Maupertuis principle of least action but recognized also in the second law of thermodynamics. This law says that energy will flow from highs to lows, for example, from hot to cold. The optimal consumption in turn is stated by the maximum entropy production principle, which says that the flow will naturally follow along least-time paths. The principle of maximum entropy production stated as "A system will select the path or assembly of paths out of available paths that minimizes the potential or maximizes the entropy at the fastest rate given the constraints"[28] does not state that the evolutionary process, due to flows of energy between the system and its surroundings, itself will affect the constraints. When the flows and differences of energy are not pronouncedly described as interdependent, the intractable character that is the essence of the present paper remains obscure in the modern statements of the old law, in contrast to the Maupertuis mathematical form. Moreover, the maximum entropy production principle in its mathematical form[29] does not explicitly specify conservation of energy in a change of state as does Eq. (1), where a change in kinetic energy balances changes in scalar and vector potential.

Energy densities, whether in stationary or evolutionary forms, are, according to the Maupertuis principle, fundamentally the same. In other words, any form of energy can in principle be converted from one to another in various changes of state due to absorptive and emissive events. Undoubtedly a particular chemical system does not have means and mechanisms to make all kinds of transformations, such as including nuclear reactions. The specific system-dependent characteristics should not obscure the general view represented by the principle of least action, which pictures all entities of reality in terms of actions. Closed actions relate to stationary densities, whereas open actions relate to the evolutionary densities. Dissipation of quanta we identify as a flow of time. Today we might talk about the bound forms of energy as the scalar potential. Invariant mass is the conserved quantity defined by Noether's theorem.[30] An open action is the energy density in propagation, that is, a boson, which is a force carrier between the fermions, as illustrated by Feynman diagrams.[31] Accordingly we associate the free forms of energy with the vector potential.

Although obvious, it is worth noting that an evolutionary path must eventually, because of the conservation of quanta, lead from one stationary state to another. During this process the closed confinement of an energy density must open up either to dissipate into, or acquire, some energy from its surroundings and then close up anew. The evolutionary step is necessarily discrete because a curve can be either closed or open but not anything in between. For example, a chemical reaction must either emit or absorb at least one quantum from its surroundings to proceed from one state to another on its way toward a stationary state. Since any closed curve is modular,[14] so must energy also be emitted or absorbed in modules of action. This quantized character of energy transduction is apparent, for example, in the photoelectric effect and in Planck's law.[32]

It is the integral confinement of energy at the initial and final state that requires that the communication between these states also be quantized. At least a quantum of action would be acquired or expelled during the evolutionary transformation from one stationary state to another. When the symmetry of one stationary state breaks up and closes again at the other end, the value of a conserved quantity, defined by Noether's invariant variation,[31] will change in steps. The evolutionary step is a topological change where a Noetherian ring will by its opening, insertion or excision, and reclosing increase or decrease its maximal element.[33] Eventually a step-by-step spontaneous symmetry-breaking process will arrive at the absolutely least closed action $\hbar$. This most elementary circulation of energy density can break open only so that momentum will spread over a wavelength to define the absolutely least open action, Planck's constant h. Its irreducible representation relates to the symmetry group U(1), whose generator is the most elementary boson known as the photon. When all other forms of energy densities at higher states of symmetry are regarded as multiples of these most elementary actions,[18,34] evolution to a natural hierarchy[35] will follow from the second law (Fig. 2). A particular assortment of actions in a system depends on the past paths and current conditions, that is, on the evolutionary history and surrounding densities. Thus the plethora of fundamental particles, as well as the diversity of biological species, can be viewed as manifestations of the quest to disperse energy in the least time. As well, we also suggest that the collection of fundamental forces is then seen only as an expression of the imperative to diminish energy density differences between the diverse forms and the surrounding vacuum. For example, energy-sparse surroundings will cause opposite charges to attract each other by accepting quanta from the high-energy density that hovers between two charges. Although electromagnetic forces of opposite charge vanish outside of net-neutral bodies, the energy density is not zero. Therefore the energy-sparse surroundings will cause two bodies to attract each other by accepting quanta from the high-energy density that exists between them. The vacuum energy density, characterized by permittivity and permeability that define the speed of light $\varepsilon_o\mu_o=c^{-2}$ and impedance $\varepsilon_o\mu_o=Z^{-2}$, manifests itself, for example, in the Casimir effect[36] and in the Aharanov-Bohm experi-

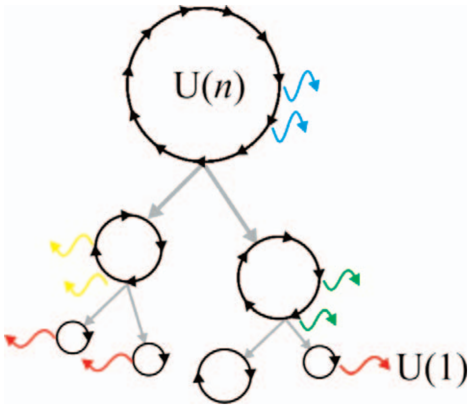

FIG. 2. (Color online) A system, depicted in its initial state as a closed modular ring at the top, evolves step by step from one stationary state of symmetry U(n) drawn as a closed circulation to another by breaking one symmetry for another when expelling quanta in the forms of open actions (wave arrows) to its surroundings. The most elementary closed action, which is the circle with a single modulus, defines an elementary locus of space. When it opens up, the transformation will yield a photon that is the generator of the symmetry group U(1).

ment.[37] Indeed, the vacuum does eject photons in the dynamic Casimir effect.[38] Furthermore the physical vacuum gives rise to the tiny but non-negligible gravitational acceleration $a_t = GM/R^2 = 1/\varepsilon_o \mu_o R = c^2 R = c/t$ due to the total mass M of the universe that has been diluting to the current average density $\rho_t = 1/2\pi G t^2$, estimated 9.9×10^{-27} kgm^{-3} from the Wilkinson Microwave Anisotropy Probe (WMAP) measurements,[39] over the huge radius R that has been expanding at the rate $H = 1/t$ during some $t = 13.7 \times 10^9$ years.[40]

V. CHARACTERISTIC COURSES

Transformations from high- to low-density actions power the universal dilution of energy densities. These ubiquitous spontaneous symmetry-breaking processes at all levels of nature's hierarchy generate sigmoid courses and skewed distributions.[41,42] At each time when a new mechanism of energy transduction emerges and taps into an intact reservoir, the stationary state will be punctuated by a burst of dissipation as evolution commences. Soon, when free energy begins to decline, the evolution will turn over to a power-law course. Eventually, when resources run out and no new mechanisms emerge, evolution will settle along an approach toward a new stasis (Fig. 3). When nonrenewable resources have been consumed, a disintegrating decline along a sigmoid course becomes inevitable.[43,44] These characteristic least-time courses of natural processes follow from the interdependency among all entities in nature. Nothing can be done without affecting something else—the essence of Newton's laws.

Often natural processes are modeled as Markovian processes. However, while this stochastic model is appropriate to describe thermodynamic stationary-state motions, it does not reveal that the free energy is the force that drives the system from one state to another. According to the principle of least action, evolution is not a random process but instead a least-time consumption of free energy. Moreover, natural systems contain a memory of the past states that may provide mechanisms to access future states. Genomes are apparent examples, but also the natural abundance of elements on Earth as well as in the entire universe contains information of past processes in various physical representations such as in the cosmic background radiation. More simply, in the stationary state even the mean of momenta encode something about global distance from equilibrium.

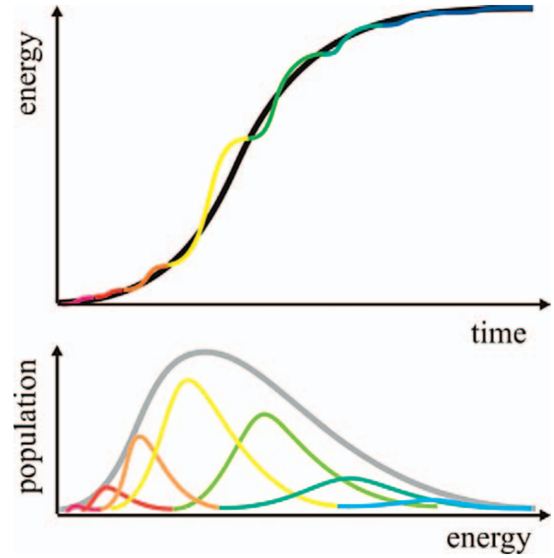

FIG. 3. (Color online) An overall natural process cumulates from a series of sigmoid growth curves (colored), each triggered by the advent of a more effective mechanism of energy dispersal (above). These mechanisms populate skewed distributions of energy densities (below). According to the law of maximal free energy consumption, the particular partition of energy transduction mechanisms (colored) of a statistical system at each state along an evolutionary path would be the most effective in producing entropy. According to Lyapunov's stability criterion[48] any deviation away from the optimal partition at any given time and place would induce gradients in energy that would be consumed by flows of energy when restoring the currently most optimal partition.

When energy flows into a system, along with available systemic constituents, it is the vital ingredient in promoting the emergence of new mechanisms with unique properties for further dispersing that energy. The rise of self-organized complexity at these critical events[45] is promoted when the assembling machinery functions more effectively in dispersing energy than what may have been there prior to the event. Since any mechanism itself is a repository of energy, it is also a potential source of consumption for other entities, such as biological species. Thus, diversity builds upon existing diversity. On the other hand, when net energy flows out of a system, its former mechanisms can no longer be assembled from the available systemic constituents, and the system must settle for less. A holistic account like this of a system and its surroundings generates the emergent characteristics of evolution, whereas reductionist decomposition techniques as well as constraint integrations are amenable only to a stationary system without net fluxes from or to the surroundings.

VII. CONCLUDING REMARKS

The principle of least action in its general mathematical form given as early as the eighteenth century by Pierre-Louis Moreau de Maupertuis, including quantization as formulated nearly 100 years ago by Emmy Noether, allows us to understand that the rules and regularities we see in nature follow from the imperative to consume free energy in the least time. Moreover, mathematical analysis of the Maupertuis action shows that these natural processes are inherently intractable when there are alternative ways to disperse energy. The flows of energy from high to low densities will themselves mold the energy-density landscape so that the circumstances for the flows keep changing. Evolution itself will change the circumstances wherever it progresses.[46,47] In computational terms, when the computation itself changes the constraints, the task will be noncomputable. It is the net flow of energy from the system to its surroundings or vice versa that provides the ingredients of emergence and renders evolution irreversible, while stationary-state dynamics are reversible and without unforeseen character in other than passing fluctuations.

This holistic outlook of nature is not new—on the contrary, many rules and regularities have been recognized in diverse disciplines and spoken of in different fields in distinct discursive languages. In the future, however, cross-disciplinary communication would benefit from initially using the most general and comprehensive concepts, and then working into particulars from that vantage point. To this end, action would function as a powerful concept since nature in its entirety and every detail can be pictured as actions. The irrevocable flows of energy from high- to low-density loci will seek, select, and then shape the paths delivering least-time energy dispersals. These ubiquitous natural processes will leave their characteristic tracks most notably in skewed distributions that populate along sigmoid curves that become dominated by power-law regions. The nondeterministic and emergent characters of these natural processes may not please our longing for predictable certainty, but in our attempts to attain a sustainable economic status we had better get acquainted up front with the most basic law of change.

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