A sidenote: Vector spaces, matrices and such

- **Vector space** $V = R^n$ (or $V = C^n$) in this chapter is defined as the set of all n-tuples (vectors) $[x_1, \ldots x_n]$ with real (or complex) entries x_i.
  - Addition of vectors and multiplication by a scalar are defined.

- Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$.

  - $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are **linearly dependent** if there is a set of scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, with at least one nonzero such that

    $$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m = 0$$

  - $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are **linearly independent** if the only set of scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, satisfiyng

    $$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m = 0$$

    is the trivial choice of

    $$\alpha_1 = \ldots = \alpha_m = 0$$

  - $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$ is a **basis** for V if for every $\mathbf{v} \in V$ there is a unique choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ for which

    $$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m$$

  - If V is a vector space with basis $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$, then every basis for V will contains exactly m vectors.
    - The number of m is called the **dimension** of V.

A sidenote: Vector spaces, matrices and such

- **Matrices** are rectangular arrays of real or complex numbers.
  - A general form of an $m \times n$ matrix is

    $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

  - Matrices have the following operations defined

    - **Addition** of two similarly shaped matrices: If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices then their sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is defined as

      $$c_{ij} = a_{ij} + b_{ij}.$$

    - **Multiplication by a scalar** $(\alpha \in R)$: $\mathbf{C} = \alpha\mathbf{A}$, $c_{ij} = \alpha a_{ij}$

    - **Multplication of two matrices**:Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{B}$ a $n \times p$ matrix. Their product $\mathbf{C} = \mathbf{AB}$ is a $m \times p$ matrix and is defined as

      $$c_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}$$

    - **Transpose** of a matrix $\mathbf{A}$, $\mathbf{C} = \mathbf{A}^T$ is defined as

      $$c_{ij} = a_{ji}.$$

    - **Conjugate transpose** of a matrix $\mathbf{A}$, $\mathbf{C} = \mathbf{A}^*$ is defined as

      $$c_{ij} = a_{ji}^*.$$

A sidenote: Vector spaces, matrices and such

- Following arithmetic equalities are easily proved:

$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A} \qquad (\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C}) \qquad \mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \qquad (\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T+\mathbf{B}^T \qquad (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

- Zero matrix has all elements zero and $\mathbf{A}+0 = 0+\mathbf{A} = \mathbf{A}$

- Unit matrix $\mathbf{1}$ of size $n \times n$ is defined as

$$\mathbf{1} = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

- For all matrices $\mathbf{A}$ of size $m \times n$ and $\mathbf{B}$ of size $n \times p$ the following apply
  $\mathbf{A1} = \mathbf{A}$, $\mathbf{1B} = \mathbf{B}$.

- Inverse of a $n \times n$ matrix $\mathbf{A}$ is defined as
  $\mathbf{AB} = \mathbf{BA} = \mathbf{1}$,
  and is denoted by $\mathbf{B} = \mathbf{A}^{-1}$.

- **Row rank** of a $m \times n$ matrix $\mathbf{A}$ is the number of linearly independent rows in it (regraded as elements in vector space R^n (or C^n)). Similarly, **column rank** is the number of linearly independent columns. These numbers are always equal.

A sidenote: Vector spaces, matrices and such

- **Vector norm** $\| \ \|$ (in vector space C^n) is defined to have the following properties:

  1) $\|\mathbf{x}\| \geq 0$, $\forall(\mathbf{x} \in C^n)$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$

  2) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$, $\forall(\mathbf{x} \in C^n, \alpha \in R)$

  3) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, $\forall(\mathbf{x}, \mathbf{y} \in C^n)$

- **Matrix norm** of a *square* $n \times n$ *matrix* $\mathbf{A}$ satisfies — in addition to 1-3 — the following:

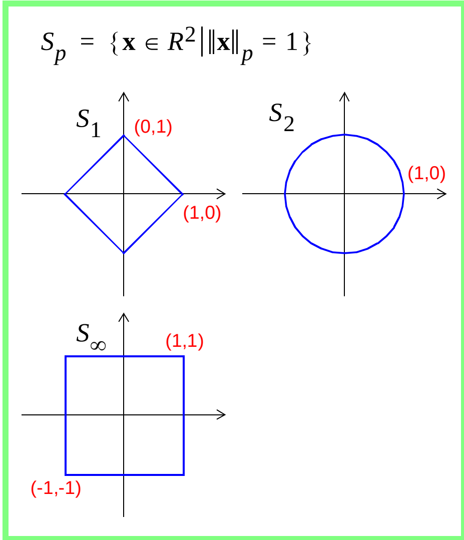
  4) $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$

  5) There exists a vector norm $\| \ \|_v$ that is *compatible* with the matrix norm:
  $\|\mathbf{Ax}\|_v \leq \|\mathbf{A}\|\|\mathbf{x}\|_v$.
  - Commonly the matrix norm used with a vector norm $\| \ \|_v$ is defined as
  $\|\mathbf{A}\| = \max_{\|\mathbf{x}\|_v = 1}[\|\mathbf{Ax}\|_v]$

A sidenote: Vector spaces, matrices and such

- Commonly used norms:

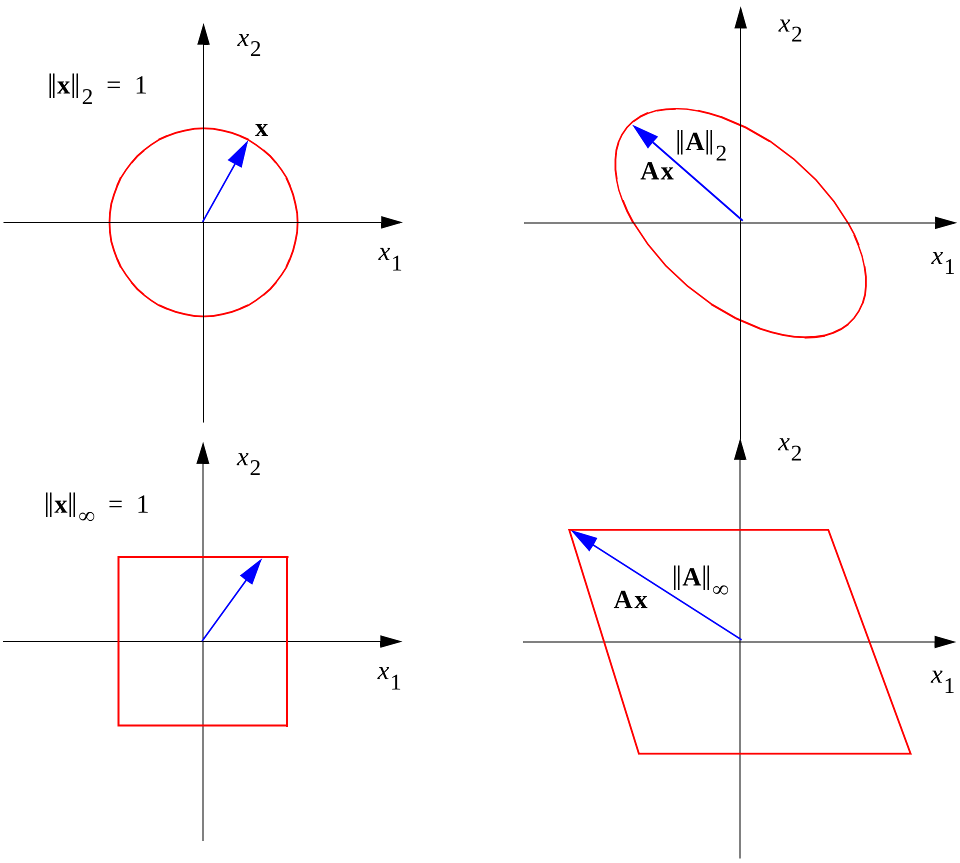
	vectors	**matrices**
p-norm:	$\|\mathbf{x}\|_p = \left[\sum_{i=1}^{n} \lvert x_i\rvert^p\right]^{1/p}$	$\|\mathbf{A}\|_p = \max_{\|\mathbf{x}\|_p = 1}[\|\mathbf{A}\mathbf{x}\|_p]$
$p = 1$	$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} \lvert x_i\rvert$	$\|\mathbf{A}\|_1 = \max_{1 \le j \le n}\left[\sum_{i=1}^{n} \lvert a_{ij}\rvert\right]$
$p = \infty$	$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n}[\lvert x_i\rvert]$	$\|\mathbf{A}\|_\infty = \max_{1 \le i \le n}\left[\sum_{j=1}^{n} \lvert a_{ij}\rvert\right]$
$p = 2$	$\|\mathbf{x}\|_2 = \left[\sum_{i=1}^{n} \lvert x_i\rvert^2\right]^{1/2}$	$\|\mathbf{A}\|_2 = \sqrt{r_\sigma(\mathbf{A}^*\mathbf{A})}$,



- Here $r_\sigma(\mathbf{M})$ is the *spectral radius* of $\mathbf{M}$; i.e. maximum size of its eigenvalues: $r_\sigma(\mathbf{M}) = \max_{\lambda \in \sigma(\mathbf{M})}[\lvert\lambda\rvert]$,
  $\sigma(\mathbf{M})$ is the *spectrum* of $\mathbf{M}$; i.e. the set of its eigenvalues
  $\mathbf{A}^*$ is the conjugate transpose of $\mathbf{A}$.
- Frobenius matrix norm $\|\mathbf{A}\|_F = \left[\sum_{i,j=1}^{n} \lvert a_{ij}\rvert^2\right]^{1/2}$ can be shown to be compatible with the vector norm $\|\mathbf{x}\|_2$.
- The following holds for a square matrix $\mathbf{A}$ and for any matrix norms of the type shown above: $r_\sigma(\mathbf{A}) \le \|\mathbf{A}\|$.

A sidenote: Vector spaces, matrices and such

- Geometric interpretation in 2D



A sidenote: Vector spaces, matrices and such

- **Rank**, **range** and **null space** of a matrix are concepts often needed in linear algebra.
  - They are defined as

let $\mathbf{A}$ be an $m \times n$ matrix

$\mathrm{ran}(\mathbf{A}) = \{\mathbf{y} \in R^m;\ \mathbf{y} = \mathbf{A}\mathbf{x};\ \mathbf{x} \in R^n\}$ (in some books also $\mathrm{range}(\mathbf{A})$; also called column space)

$\mathrm{rank}(\mathbf{A}) = \dim(\mathrm{ran}(\mathbf{A}))$ (same as the row or column rank above)

$\mathrm{null}(\mathbf{A}) = \{\mathbf{x} \in R^n;\ \mathbf{A}\mathbf{x} = 0\}$