## *INTRODUCTION:*

- Duality arguments suggest that the superconductor phase transition is in the 3d XY model universality class, but with inverted temperature [1]. Thus, XY model symmetric (broken) phase $\leftrightarrow$ normal (superconducting) phase of the superconductor.

- Duality predicts that the *Abrikosov vortex tension* $\mathcal{T}$ scales with the XY model exponent $\nu_{\rm XY} = 0.6723$. The *penetration depth* $\lambda$ (or inverse photon mass) is also argued to scale with exponent $\nu' = \nu_{\rm XY}$.

- However, universality has *not* been unambiguously observed:
  - High-$T_c$ $YBa_2Cu_3O_{7-\delta}$ experiments [3, 4] observe $\nu' \approx 0.3 \ldots 0.5$
  - Monte Carlo simulations of the Ginzburg-Landau theory appear to favour $\nu' \sim 0.3$ [5].

- A problem for the Monte Carlo simulations is that the duality is expected to be valid only in a very narrow temperature range around the critical temperature.

- In order to avoid the problem above, we study a special limit of Ginzburg-Landau theory, the *frozen superconductor (FZS)* (an integer-valued gauge theory), which is *exactly* dual to the XY-Villain model at all temperatures. Thus, the transition in FZS is bound to be in the XY model universality class. Studying the critical quantities of FZS can shed light on the problems faced in the Ginzburg-Landau theory simulations. Detailed results are published in [2].

## *MODELS:*

We start from the lattice Ginzburg-Landau theory in the London limit:

$$\mathcal{L}_{\mathrm{GL}} = \frac{1}{2}\sum_{i<j} F^2_{\vec{x},ij} + \kappa \sum_i s\left(\theta_{\vec{x}+i} - \theta_{\vec{x}} - qA_{\vec{x},i}\right)$$

Here $A_{x,i}$ is a real-valued gauge field, $\theta_x$ is the spin angle variable, and $F_{\vec{x},ij}$ is the (non-compact) plaquette. Usually $s(x) = -\cos(x)$, but we will use the Villain form

$$s(x) = -\ln \sum_{k=-\infty}^{\infty} \exp\left(-\frac{1}{2}(x - 2\pi k)^2\right).$$

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We shall study the following 2 limiting cases of the GL model:

### *Frozen Superconductor (FZS):*

Take $\kappa \to \infty$ and define $\beta = 4\pi^2/q^2$ in the GL model $\mapsto$

$$Z_{\mathrm{FZS}}(\beta) = \sum_{\{I_{\vec{x},i}\}} \exp\left(-\frac{\beta}{2}\sum_{\vec{x},i>j} \square^2_{\vec{x},ij}\right),$$

with $\square_{\vec{x},ij} = I_{\vec{x},i} + I_{\vec{x}+i,j} - I_{\vec{x}+j,i} - I_{\vec{x},j}$ and the link variables $I_{\vec{x},i}$ take integer values.

### *XY Model:*

Take $q \to 0$ in the GL model $\mapsto$ XY model with the Villain action

$$Z_{\mathrm{XY}}(\kappa) = \int D\theta \exp\left(-\kappa \sum_{\vec{x},i} s(\theta_{\vec{x}+i} - \theta_{\vec{x}})\right).$$

## *DUALITY:*

The frozen superconductor and the XY-Villain model are *exactly dual* to each other with the identification $\beta = 1/\kappa$, i.e.

$$Z_{\mathrm{XY}}(\kappa) = Z_{\mathrm{FZS}}(\beta = \frac{1}{\kappa})$$

*Proof:*

Introduce a real vector field $h_{\vec{x},i}$. Now we can write the XY model partition function as

$$Z_{\mathrm{XY}}(\kappa) \propto \int D\theta Dh_i \sum_{k_i} \exp\left[-\sum_{\vec{x},i}\left(\frac{1}{2\kappa}h_{\vec{x},i}^2 - ih_{\vec{x},i}\Delta_{\vec{x},i}^{\mathrm{XY}}\right)\right],$$

where $\Delta^{\mathrm{XY}}$ is the Noether current

$$\Delta_{\vec{x},i}^{\mathrm{XY}} = \theta_{\vec{x}+i} - \theta_{\vec{x}} - 2\pi k_{\vec{x},i}.$$

Integrating over $\theta$ yields a delta function $\delta(\vec{\nabla}\cdot\vec{h})$, where we have defined the lattice divergence

$$\vec{\nabla}\cdot\vec{h}_{\vec{x}} = \sum_i \left(h_{\vec{x},i} - h_{\vec{x}-i,i}\right).$$

The summation over $k_{\vec{x},i}$ restricts $h_{\vec{x},i}$ to integer values, and we obtain

$$Z_{\mathrm{XY}}(\kappa) \propto \sum_{\{h\}} \delta_{\vec{\nabla}\cdot\vec{h},0} \exp\left(-\frac{1}{2\kappa}\sum_{\vec{x},i} h_{\vec{x},i}^2\right),$$

which is the partition function of an integer-valued and sourceless vector field. In an infinite volume, we can interpret the vector field $h_{\vec{x},i}$ as the integer valued flux through the dual lattice plaquette pierced by link $(\vec{x},i)$, and write

$$h_{\vec{x},i} = \frac{1}{2}\epsilon_{ijk}\Box_{\vec{x},jk},$$

which implies $Z_{\mathrm{XY}}(\kappa) \propto Z_{\mathrm{FZS}}(1/\kappa)$ $\Box$

## *PHASE STRUCTURE:*

The XY-Villain model has a symmetry breaking transition at $\kappa = \kappa_c \approx 0.333068(7)$ [2]. Because of the exact duality, FZS must have a transition at $\beta_c = 1/\kappa_c$, which is of XY model universality.

The phases of the models are related by duality:

| XY model: | $\leftrightarrow$ | FSZ |
|---|---|---|
| symmetric phase $\kappa < \kappa_c$ | $\leftrightarrow$ | superconducting phase $\beta > \beta_c$ |
| broken phase $\kappa > \kappa_c$ | $\leftrightarrow$ | Coulomb phase $\beta < \beta_c$ |

## *CRITICAL OBSERVABLES:*

The duality relates the physical observables in the frozen superconductor to dual observables in the XY model.

- *Vortex tension*

  The duality relation implies that the XY model scalar correlation function equals the FSZ Abrikosov-Nielsen-Olesen "vortex operator." Especially, the XY model scalar mass $m = \mathcal{T}$, the vortex tension in FZS, and

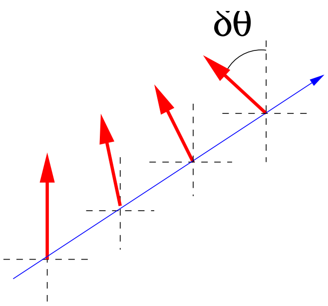
| | |
|---|---|
| $m = \mathcal{T} \propto \lvert\kappa - \kappa_c\rvert^{\nu_{\rm XY}}$ | symmetric/superconducting phase |
| $m = \mathcal{T} = 0$ | broken/Coulomb phase |

  The (well-known) value of the critical exponent is $\boxed{\nu_{\rm XY} \approx 0.6723}$.

- *Magnetic permeability*

Duality relates the *FZS magnetic permeability* $\chi_m$ to the *helicity modulus* $\Upsilon$ of the XY model.

XY model helicity modulus $\Upsilon$ is the response of the system to a "twist" of the spins along, say, 3-direction (usually implemented by boundary conditions)




$$\Upsilon = \frac{L_3}{L_1 L_2} \left( \frac{\partial^2 F}{\partial(\delta\theta)^2} \right)_{\delta\theta=0}$$

FZS magnetic permeability: susceptibility of magnetic flux to 3-direction $\chi_m = \langle \left( \sum_{\vec{x}} \Box_{\vec{x},12} \right)^2 \rangle$.

Duality again implies $\Upsilon = \chi_m$, and

| | |
|---|---|
| $\Upsilon = \chi_m = 0$ | symmetric/superconducting phase |
| $\Upsilon = \chi_m \propto \lvert\kappa - \kappa_c\rvert^{\upsilon}$ | broken/Coulomb phase |

It has been argued that the critical exponent $\boxed{\upsilon = \nu_{\mathrm{XY}}}$.

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- *Gauge field susceptibility*

$\chi_m$ vanishes in the superconducting phase. However, we can define the gauge field susceptibility $\chi_A$ by

$$\chi_A = \lim_{\vec{p} \to 0} \frac{\Gamma_{ii}(\vec{p})}{\vec{p}^2} \,,$$

where $\Gamma_{ii}$ is the (diagonal) photon correlation function. This diverges as $\beta \longrightarrow \beta_c+$ as

$$\boxed{\chi_A \propto (\beta - \beta_c)^{-\nu_A} \quad \text{superconducting phase } \beta > \beta_c}$$

and the duality implies that $\boxed{\nu_A = \nu_{\mathrm{XY}}}$.

- *Photon mass*

The most natural observable for the FSZ model (and the one usually measured in high-$T_c$ superconductor experiments) is the photon mass $m_\gamma = 1/\lambda$, the inverse of the *penetration depth*. The duality relates $\lambda$ to the correlation length of the XY-Villain model current operator

$$\Delta^{\mathrm{XY}}_{\vec{x},i} = \theta_{\vec{x}+i} - \theta_{\vec{x}} - 2\pi k_{\vec{x},i}.$$

Parametrizing the critical behaviour of $m_\gamma$ with the exponent $\nu'$, we have

| | |
|---|---|
| $m_\gamma = 1/\lambda \propto \lvert\beta - \beta_c\rvert^{\nu'}$ | superconducting phase $\beta > \beta_c$ |
| $m_\gamma = 1/\lambda = 0$ | Coulomb phase $\beta < \beta_c$ |

The theoretical predictions for $\nu'$ have varied in the range $\nu' \approx 0.3 \ldots 0.5$, before settling down to $\boxed{\nu' = \nu_{\mathrm{XY}}}$ [6].

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- *Anomalous dimension of the gauge field*

The anomalous dimension $\eta_A$ of the gauge field at criticality can be obtained from the momentum dependence of the susceptibility

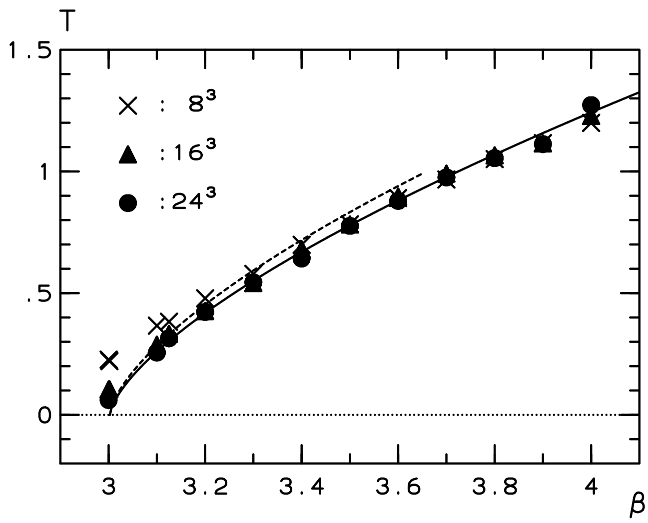
$$\chi_A(p) = \frac{\Gamma_{ii}(\vec{p})}{\vec{p}^2} \propto (\vec{p}^2)^{\eta_A/2-1} \qquad \beta = \beta_c$$

Theoretical prediction for the anomalous dimension is $\eta_A = 1$ [6] (also verified in earlier Monte Carlo simulations).

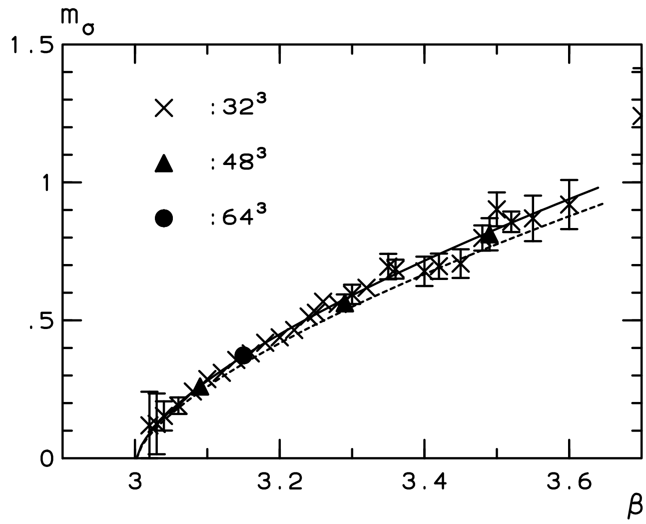
# *MEASUREMENTS*

## *Vortex tension*

We measure the vortex tension $\mathcal{T}$ in the FSZ using multicanonical simulations and special boundary conditions, which smoothly extrapolate from $0 \mapsto 1$ vortices on the lattice (for details, see [2]). The results in the superconducting phase $\beta > \beta_c$ are shown below, left.




Vortex tension in FSZ

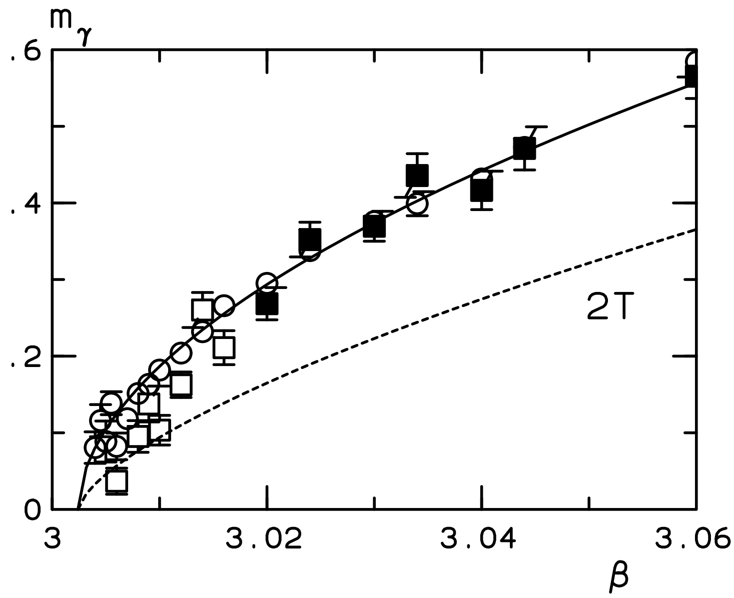



Scalar mass in the XY model

The scalar mass $m$ in the XY-Villain model is shown on the right, plotted against $\beta = 1/\kappa_{\rm XY}$. The continuous lines are power law fits, and, for comparison, the dashed lines show the fits transferred from the other plot. As predicted by the duality, $\mathcal{T} = m$ within the statistical errors. The critical exponent of the tension fit is $\nu_{\mathcal{T}} = 0.672(9)$, compatible with $\nu_{\mathcal{T}} = \nu_{\rm XY}$.

## *Photon mass*

We have measured the photon mass $m_\gamma$ from the plaquette-plaquette correlation functions in FSZ. The results in the superconducting phase, in very close proximity to the critical point, are shown below:



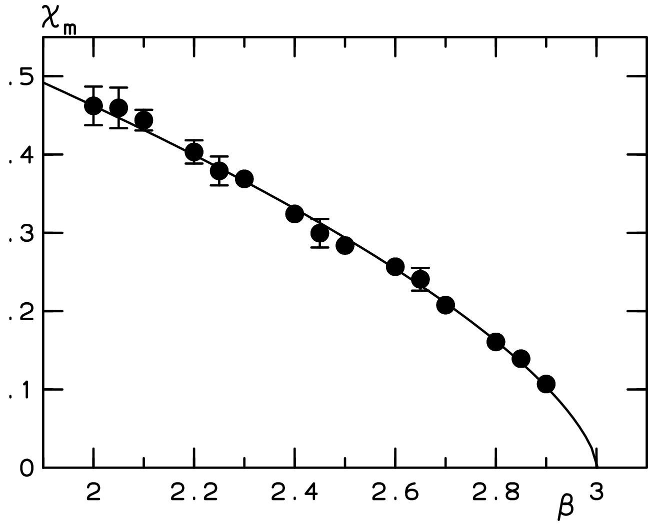

A power law fit to data yields $m_\gamma \propto (\beta - \beta_c)^{\nu'}$, with $\nu' = 0.54(6)$. This agrees with the experimental observation in [4] and with the mean-field theory, but *is not compatible with the duality prediction* $\nu' = \nu_{\rm XY}$.

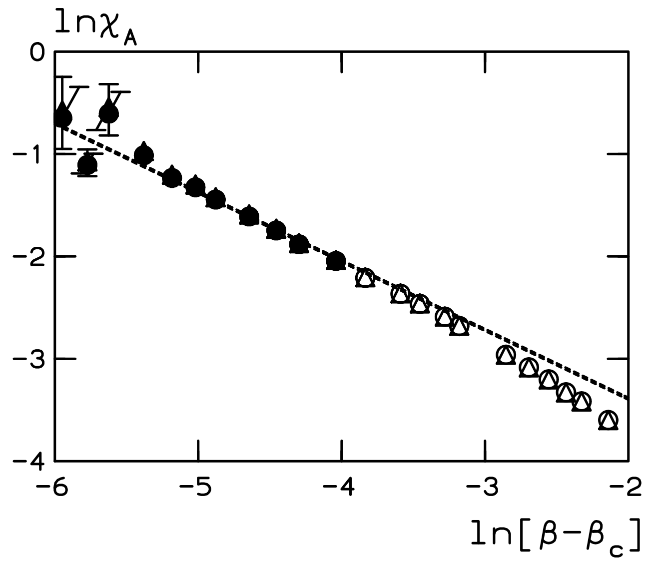
The dashed line shows $2 \times \mathcal{T}$, the vortex tension. Since the photon couples with two vortices, and the measured $m_\gamma > 2 \times \mathcal{T}$, we conclude that the *measured* $m_\gamma$ *shows only pre-asymptotic behaviour*, and the true asymptotic photon channel correlation must be

$$m_\gamma = 2 \times \mathcal{T} \propto (\beta - \beta_c)^{\nu_{\rm XY}}$$

which agrees with the prediction from the duality.

## *Magnetic permeability and gauge field susceptibility*



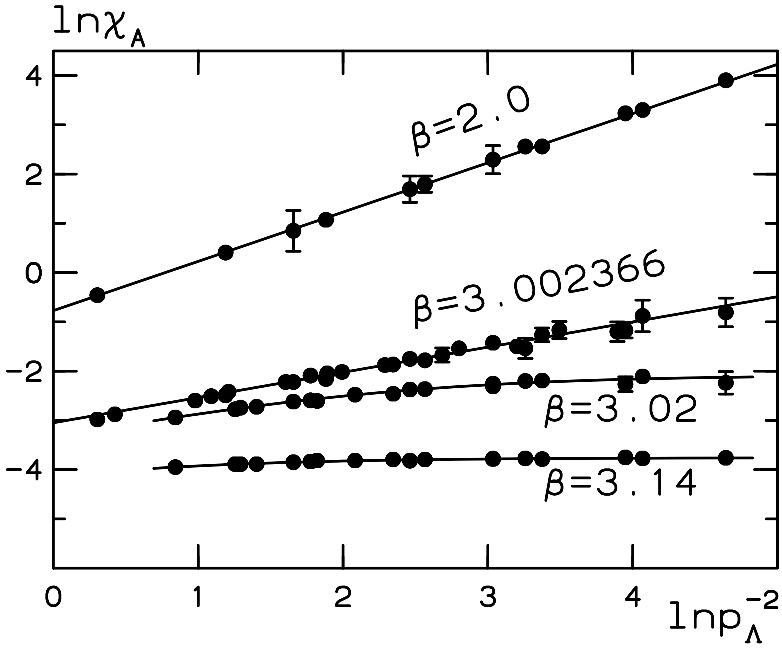




Magnetic permeability $\chi_m$ gauge field susceptibility $\chi_A$

The plot on the left shows the magnetic permeability $\chi_m$ of FZS, measured in the Coulomb phase $\beta < \beta_c$. Power law fit gives the critical exponent $v = 0.66(2)$, which is perfectly compatible with the duality prediction $v = \nu_{\mathrm{XY}}$.

The superconducting phase ($\beta > \beta_c$) gauge field susceptibility $\chi_A$ is shown on the right on a log-log plot. Clearly, $\chi_A$ diverges with a power law as $\beta_c$ is approached; a fit at very close proximity to $\beta_c$ yields the critical exponent $\nu_A = 0.69(4)$, which again agrees with the duality prediction $\nu_A = \nu_{\mathrm{XY}}$.

## *Anomalous dimension of the gauge field*




The plot shows the momentum dependence of the susceptibility $\chi_A(p)$ at the criticality ($\beta = \beta_c \approx 3.002366$), in the superconducting phase ($\beta > \beta_c$) and in the Coulomb phase ($\beta < \beta_c$). In the Coulomb phase $\chi_A(p) \propto p^{-2}$, and in the superconducting phase $\chi_A(p)$ remains finite as $p \to 0$.

At the critical point the susceptibility scales with anomalous dimension: $\chi_A(p) \propto |\vec{p}|^{\eta_A - 1/2}$. The power law fit gives $\eta_A = 0.98(4)$, which is perfectly consistent with predictions [6].

## *CONCLUSIONS:*

- Frozen superconductor (FSZ) is a limiting case of the Ginzburg-Landau theory, which is *exactly dual* to the XY model with the Villain action.
- Because of the exact duality, the Coulomb-superconductor phase transition in FSZ must be in the XY model universality class.
- We have measured several critical observables in FSZ, and duly find the behaviour predicted by XY model duality for most of the quantities.
- The exception is the *photon mass* $m_\gamma$ (inverse penetration depth), which appears to scale with an exponent which is substantially smaller than the XY model prediction. This has also been observed in high-$T_c$ superconductor experiments and lattice studies of the Ginzburg-Landau theory.
- However, since the photon couples to 2 vortices, and $m_\gamma > 2 \times \mathcal{T}$, the photon can decay into 2 vortices. Thus, $m_\gamma$ *must* scale with the same exponent as the string tension $\mathcal{T}$, which is the scaling behaviour predicted by duality.
- The apparent incorrect scaling behaviour may be due to the large anomalous exponent of the photon correlation function, $\eta_A = 1$.

$\longrightarrow$ The photon mass (or the penetration depth) appears to be singularly difficult observable for determining the critical behaviour, also in superconductor experiments.

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