QCD Equation of state in perturbation theory (and beyond. . .)

Kari Rummukainen

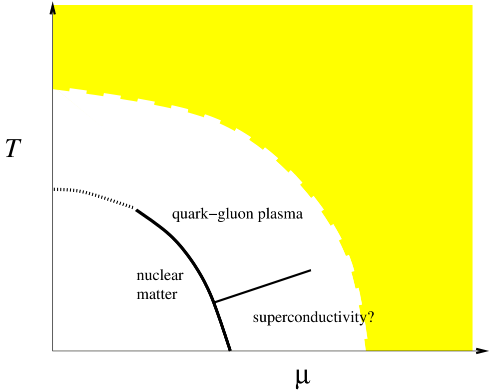
CERN Theory & University of Oulu, Finland

Physics of high baryon density

"QCD is perturbative if $T, \mu \gg \Lambda_{QCD}$"

- Weakly interacting quark-gluon plasma, with $g^2 \ll 1$
- **True:** $\exists$ Asymptotic coupling constant expansion:

$$p/T^4 = g^0 + g^2 + g^3 + g^4 + \ldots$$



- **False:** Coefficients of the expansion are *not calculable* in loop expansion beyond some low order, due to infrared singularities. For pressure, this happens at order g^6: 3-dim confinement! [Linde 1980]
- Perturbation theory and lattice nicely complement each other

Status (pressure = free energy):

- p/T^4 can be calculated only up to $O(g^6 \ln 1/g)$ in P.T.
- $p(T, \mu = 0)$ fully known [Kajantie, Laine, K.R., Schröder 02]
- $p(T, \mu_i)$ also known fully ($T \gtrsim g\mu$?) [Vuorinen 03]
- $p(T = 0, \mu_i)$ known to order $O(g^4)$ [Freedman, McLerran 77]
- $p(T, \mu_i)$ for all T known to order $O(g^4)$ [Ipp, Kajantie, Rebhan, Vuorinen 06]
- To go beyond $O(g^6 \ln 1/g)$ requires *non-perturbative* input (3-dim lattice, partly done ...)
- **Bad convergence for $T \lesssim 100\, T_c$!**

$\mapsto$ Far from ideal gas (Stefan-Bolzmann)

- **Still missing:** all of the above is with $m_q = 0$!

"Resummed" perturbation theory

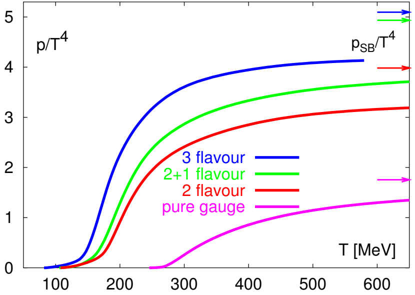
- Bad convergence in std. expansion $\mapsto$ reshuffle to improve
- – Screened P.T. (not suitable for QCD) [Karsch, Patkos, Petreczky]
  – Hard Thermal Loop P.T. [Andersen, Braaten, Petitgirard, Strickland]
  – "Φ-derivable" approach [Blaizot, Iancu, Rebhan; Peshier]
- Aim: improved convergence at larger couplings (low T)
- Typically include *quasiparticles, Debye screening, damping* etc. from the outset: more physical thermal "vacuum" to expand around?
- Reproduce std. perturbative expansion to some (low) order, depending on truncations, but include contributions from all higher orders.
- Non-perturbative magnetostatic sector (g^6, 3d confinement), is *not included*. If this is important these approaches fail (as does normal P.T.)

Why precise equation of state at $T > T_c$ is desirable?

- Cosmology: precision measurements require $\lesssim$1% level accuracy in EoS
- HIC: input for hydro; susceptibilities
- CBM: boundary condition for models(?)
- plain theoretical interest

Lattice QCD

Standard lattice QCD works very well when $T \lesssim 5\text{–}10\, T_c$:



[Karsch 2001]

- Pure gauge $N_f = 0$: $p(T \leq T_c) \approx 0$ since glueballs heavy
- $N_f = 2, 3$: at $T < T_c$ gas of (light) pions
- $p'(T_c)$ discontinuous $\Rightarrow$ genuine 1st order phase transition ($\exists$ for pure gauge QCD)
- Lattice QCD runs out of steam at $T \sim 5\text{–}10\, T_c$! Hierarchy of scales $\Lambda_{\rm QCD} \ll T \ll 1/a$.
- Relation to perturbation theory?

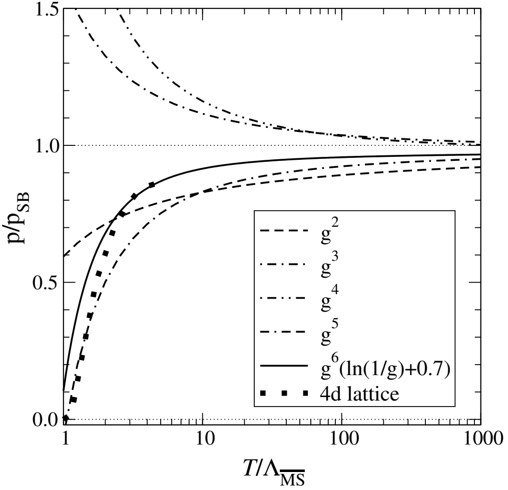
Pressure in perturbation theory

The perturbative expression for the pressure is fully known ($m_q = 0$):

$p/p_{\rm SB}$	$= 1$	Stefan-Boltzmann	
	$+g^2$	2-loop	[Shuryak 78]
	$+g^3$	resummed 2-loop	[Kapusta 79]
	$+g^4 \ln 1/g$	resummed 2-loop	[Toimela 83]
	$+g^4$	resum 3-loop	[Arnold, Zhai 94]
	$+g^5$	resum 3-loop	[Kastening, Zhai 95]
	$+g^6 \ln 1/g$	resum 4-loop	[Laine, Kajantie, K.R., Schroede
	$+g^6$	not computable!	[Linde 80]
	$+g^7 + g^8$	computable	
	$+g^9 + g^{10} + \ldots$	not computable	

$$p_{\rm SB} = \frac{\pi^2 T^4}{45}\left(8 - \frac{21 N_f}{4}\right)$$

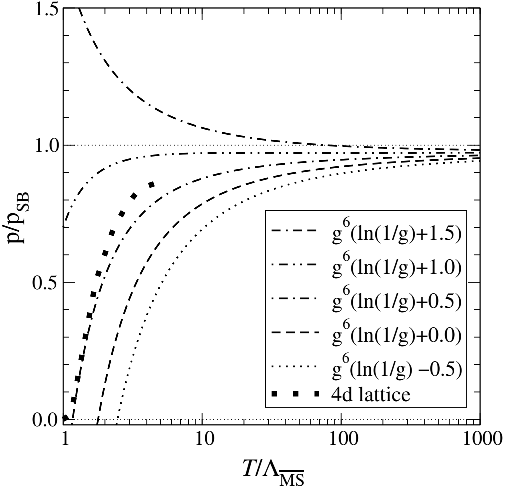
P.T. alone does not provide the answer:



[Laine, Kajantie, K.R, Schroeder 02]

- Generally bad convergence

Dependence on g^6-coefficient:



- Non-perturbative effects [parametrized here with $g^6 \times (const.)$] can be very significant!

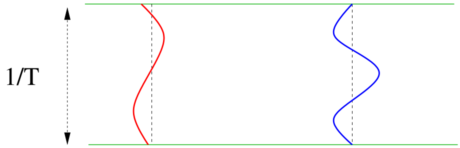
Loops at finite T

Finite T ensemble: euclidean with imaginary time extent $1/T$, with (anti)periodic b.c for bosons (fermions) $\Rightarrow$

$$\frac{1}{p^2} \longrightarrow \frac{1}{\bar{p}^2 + \omega_n^2}, \qquad \omega_n = \begin{cases} 2n\pi T & n \in Z \quad \text{Bosons} \\ (2n+1)\pi T & n \in Z \quad \text{Fermions} \end{cases}$$

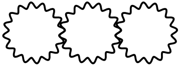
$$\int \frac{d^4 p}{(2\pi)^4} \longrightarrow T \sum_n \int \frac{d^3 \vec{p}}{(2\pi)^3}$$

Thus, all $n \neq 0$ Bosonic modes and all Fermionic modes acquire a "mass" $\sim \pi T$. Clearly, only bosonic $n = 0$ modes are infrared sensitive.



Why non-analytic in g^2?

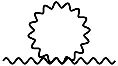
Consider 3-loop contribution to the pressure (superficially g^4), and $n = 0$ modes. Now the middle loop is IR divergent:



$$\sim T \sum_n \int \frac{d^3\vec{p}}{[\vec{p}^2 + (2\pi n T)^2]^2} \xrightarrow{n=0} \int_0^\infty \frac{dp\, p^2}{[p^2]^2}.$$

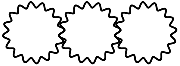
Need to resum. Consider 1-loop polarization (of *electric* gluon modes)

$$m_D^2 \sim$$



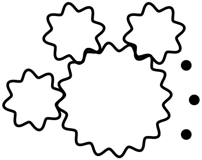
$$\sim g^2 T \sum_n \int \frac{d^3\vec{p}}{\vec{p}^2 + (2\pi n T)^2} \sim g^2 T^2 + \text{UV-divergent}$$

Electric modes are screened by Debye mass m_D. Insert m_D on all propagators:



$$\sim [g^2 T^2]^2\, T \sum_{n=0} \int \frac{d^3\vec{p}}{[\vec{p}^2 + m_D^2]^2} \sim g^4 T^5 / m_D \approx g^3 T^4$$

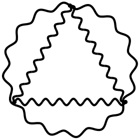
Effectively, this means that we need to (re)sum over all diagrams of type



All of these contribute to order g^3!

Why $g^6 T^4$ is non-perturbative?

Debye mass m_D does not regulate magnetic sector!
Let us consider vacuum diagram at finite T, where we add a (fictitious) mass term m to keep track of the scale: [Linde 80]



N loops $\rightarrow \begin{cases} (N-1) & \text{4-vertices} \\ (2N-2) & \text{propagators} \end{cases}$

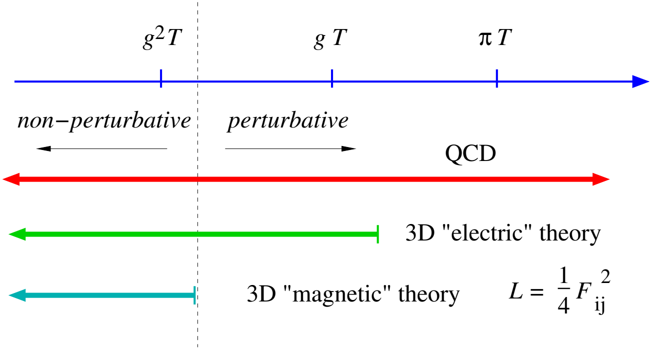
$$\left[T \int d^3p\right]^N (g^2)^{N-1} \left[\frac{1}{q^2 + m^2}\right]^{2N-2} \propto g^6 T^4 \left[\frac{g^2 T}{m}\right]^{N-4}$$

If $m = g^2 T$ (“magnetic” scale), all loops contribute to pressure at g^6!
Perturbatively $m = 0$ for magnetic gauge modes $\Rightarrow$
Loop expansion fails when $\boxed{k \lesssim g^2 T}$ (3d confinement).
Note: Loop expansion OK when
$k \sim \pi T$ (hard scales, $n \neq 0$)
$k \sim gT$ (Debye, electric scales)

3d effective theories

An efficient way to implement resummations and organize the perturbative (and lattice) calculations

Energy scales:



Hierarchy of effective theories [Braaten,Nieto 95]

- Integrate over πT (fermions!) $\Rightarrow$ 3-dim. effective theory $\mathcal{L}_{\mathrm{E}}$ for gT, $g^2 T$ -modes
- Integrate over $gT \Rightarrow$ 3-dim. effective theory $\mathcal{L}_{\mathrm{M}}$ for $g^2 T$ -modes

("integrate" $\equiv$ 2-loop optimized matching of theories)

Electric effective theory $\mathcal{L}_\mathrm{E}$ is 3d adjoint Higgs model:

$$\boxed{\mathcal{L}_\mathrm{E} = \frac{1}{2}\mathrm{Tr}\,F_{ij}^2 + \mathrm{Tr}[D_i, A_0]^2 + m_D^2\,\mathrm{Tr}\,A_0^2 + \lambda_A(\mathrm{Tr}\,A_0^2)^2}$$

For $N_f = 0$ (pure gauge), $\mu = 0$ the couplings are

$$\begin{aligned} g_3^2 &= g^2 T = \frac{8\pi^2}{11\ln(6.742\,T/\Lambda_{\overline{\mathrm{MS}}})}\,T \\ m_D^2 &\sim g^2 T^2 \\ \lambda_A &\sim g^4 T \end{aligned}$$

$\mathcal{L}_\mathrm{E}$ can be analyzed using perturbation theory or lattice simulations. Pressure is accurate to order g^6!

Magnetic $\mathcal{L}_\mathrm{M}$ is just 3d QCD: $\boxed{\mathcal{L}_\mathrm{M} = \frac{1}{2}\mathrm{Tr}\,F_{ij}^2}$

with coupling constant $g_3^2 \approx g^2 T$.

Pressure:

$$\frac{p}{p_{\mathrm{SB}}} = 1 + g^2 + \underbrace{\overbrace{g^3 + g^4 \ln\frac{1}{g} + g^4 + g^5 + g^6 \ln\frac{1}{g}}^{\longleftarrow\ \textit{pert. theory}}}_{\mathcal{L}_{\mathrm{E}} \longrightarrow} + \underbrace{g^6}_{\mathcal{L}_{\mathrm{M}}} + g^7 + \ldots$$

The relation between physical pressure and 3d free energy is [(Braaten,Nieto)]

$$\frac{p}{p_{\mathrm{SB}}} = 1 - \frac{5}{2}\frac{\lambda_A}{g_3^2} - \frac{45}{8\pi^2}\left(\frac{g_3^2}{T}\right)^3 (\mathcal{F}_E + (\mathrm{known}))$$

where

$$\mathcal{F}_E = -\frac{1}{V g_3^6} \ln \int [dA] \exp[-\int d^3x \mathcal{L}_{\mathrm{E}}]$$

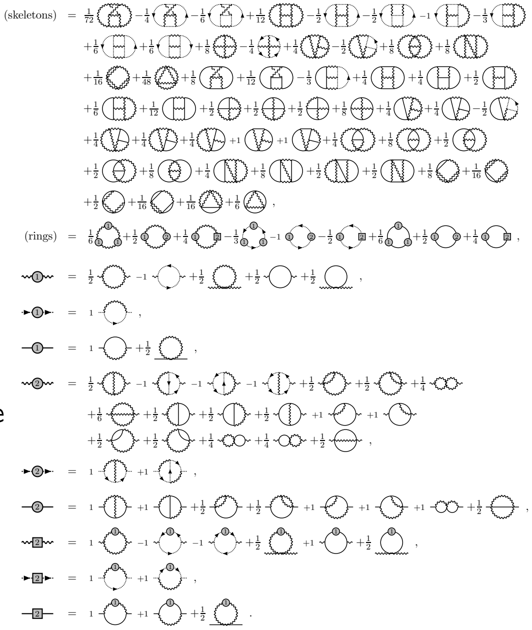
$\mathcal{F}_E$ can now be calculated
– perturbatively $\mapsto g^6 \ln \frac{1}{g}$
– non-perturbatively $\mapsto$ higher order

4-loop graphs for $g^6 \ln \frac{1}{g}$:

$\#$ of diagrams $\sim$
3:6:47:490 at 1:2:3:4 loops

One 4-loop diag. with 6 3-point vertices and 9 propagators contains $\sim 24 \times 10^6$ terms, to be integrated over!
$(2^9 \times (3 \times 2)^6)$

Computer algebra program is a *must* to sort out the integrals (FORM)



Non-perturbative g^6-contribution

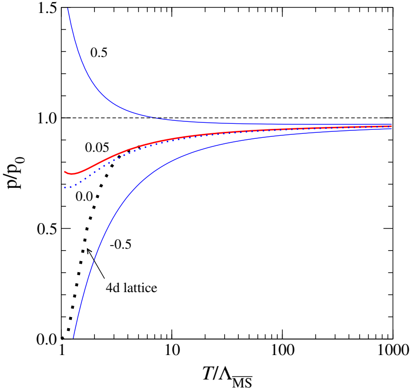
- g^6 coefficient is of interest: first genuinely non-perturbative coefficient; can have significant quantitative effect.
- The cleanest way to obtain the genuinely 3-dimensional $g^6 T^4$ contribution is to use $\mathcal{L}_{\mathrm{M}}$, i.e. 3d pure gauge QCD. It is directly related to the condensate $\langle F_{ij}^2 \rangle$. [Braaten, Nieto 95; Karsch, Lütgemeier, Patkos, Rank 96]
- This can be determined on the lattice. [Hietanen et al. 04]
- However, this requires 4-loop lattice↔continuum matching:

$$\begin{aligned} \langle F_{ij}^2 \rangle_{\mathrm{lat}} &= \frac{1}{a^3} + \frac{g^2 T}{a^2} + \frac{(g^2 T)^2}{a} \\ &+ (g^2 T)^3 (\ln \frac{1}{g^2 T a} + [\text{matching coeff.}] + [\text{non-pert. physics}]) \end{aligned}$$

- This has not been done yet (stochastic perturbation theory?)

Lattice simulations of $\mathcal{L}_E$

- Lattice simulations of the electric theory $\mathcal{L}_E$ give information about higher order contributions: $g^7 + g^8 + \ldots$



[Kajantie, Laine, K.R., Schroeder 01]

- g^6 has to be tuned $\rightarrow$ matching to 4d lattice
- Not good enough accuracy yet

Hard Thermal Loop Perturbation Theory

- Lagrangian:

$$\mathcal{L} = \left[\mathcal{L}_{\rm QCD} + (1-\delta)\mathcal{L}_{\rm HTL}\right]_{g^2\to\delta g^2}$$

- $\delta = 1$ is QCD
- $\mathcal{L}_{\rm HTL}$ is constructed so that already the free propagator ($\delta = g^2 = 0$) gives thermal quasiparticles, screening (m_D), Landau damping ...

$\mapsto$ more physical starting point, better behaved expansion

- Aim: bring validity of expansion close to T_c
- Calculations technically very difficult already at 2-loop level!

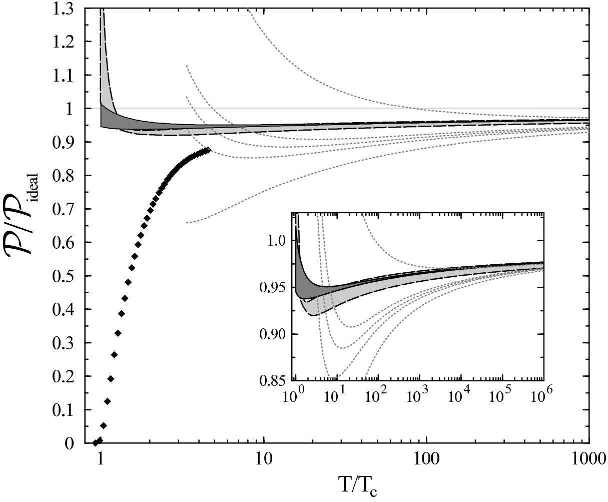
$N_f = 0$ QCD with HTLPT at 1 (light gray) and 2 (dark gray) loops:

Compared with 4d lattice (dots) and 3d lattice ($\mathcal{L}_E$, dotted lines)

Good convergence, overshoots the 4d lattice result

Formally accurate to order $g^4 \ln 1/g$

[Andersen, Braaten, Petitgirard, Strickland 02]



Conclusions

- P.T. has been pushed as far as it goes (for $p(T, \mu)$)
- Slow convergence; large order-by-order fluctuations

$\Rightarrow$
  - System strongly coupled
  - Not close to Stefan-Bolzmann
    ($10\,T_c: \ \sim 12\%, \ \ 100\,T_c: \ \sim 7\%$)
  - Large *perturbative* deviations from ideal gas (can also be non-perturbative)

- To be done:
  - g^6 non-perturbative contribution
  - $m_q \neq 0$