Pressure in hot QCD – how non-perturbative is it?

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- ***"QCD is perturbative when*** $T, \mu \gg \Lambda_{QCD}$***"***

$\Rightarrow$ Weakly interacting gas of quarks and gluons, *quark-gluon plasma*

- Can apply perturbation theory
- But: *infrared singularities* $\Rightarrow$ perturbation theory breaks down at finite order! Linde 1980

- Pressure $p = -F/V$ at $T \gtrsim 4T_c$
- This is where standard lattice QCD data stops
- Use *3d effective theory*
  – explicit Lat $\leftrightarrow$ $\overline{\mathrm{MS}}$ connection
  – use P.T. as much as you can (all calculable contributions)
  – rest non-perturbatively
- Heavy-Ion collisions, Cosmology, ...
- Resummation techniques

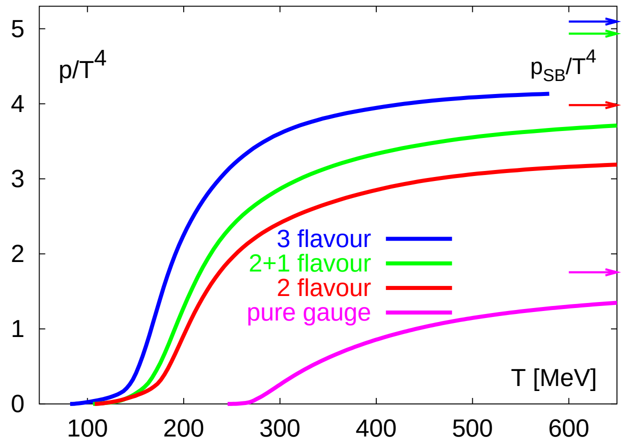
  Andersen, Braaten, Petitgirad, Strickland;

  Blaizot, Iancu, Rebhan;

  Parwani, Peshier, Cvetic, Kögerler ...

Lattice QCD

Standard lattice QCD works very well when $T \lesssim 5$–$10T_c$:



Karsch 2001

- Pure gauge $N_f = 0$: $p(T \leq T_c) \approx 0$ since glueballs heavy
- $N_f = 2, 3$: at $T < T_c$ gas of (light) pions
- $p'(T_c)$ discontinuous $\Rightarrow$ genuine 1st order phase transition ($\exists$ for pure gauge QCD)

Lattice QCD runs out of steam at $T \sim 5$–$10T_c$!

- Free energy on a lattice
  $$e^{-fV/T} = \int [dU]\, e^{-S[U]}$$
  is finite and (in principle) calculable.
- However, it is UV divergent ("cosmological constant").
- Solution: subtract f from $T = 0$ lattice
  $$p_{\text{phys.}}(T) = -\lim_{a \to 0} [f_{\text{lat}}(T, a) - f_{\text{lat}}(T = 0, a)]$$
- Lattice spacing $a \ll 1/T$
- $T = 0$ means size of the lattice $L \gg 1/T_c \sim 1/\Lambda_{\text{QCD}}$
- Wide range of scales: $a \ll 1/T \ll 1/T_c \ll L$, all included in $T = 0$ lattice! Very expensive at high T.

– Still a substantial difference to the free Stefan-Bolzmann behaviour.

– What is the relation to perturbation theory?

Pressure in perturbation theory

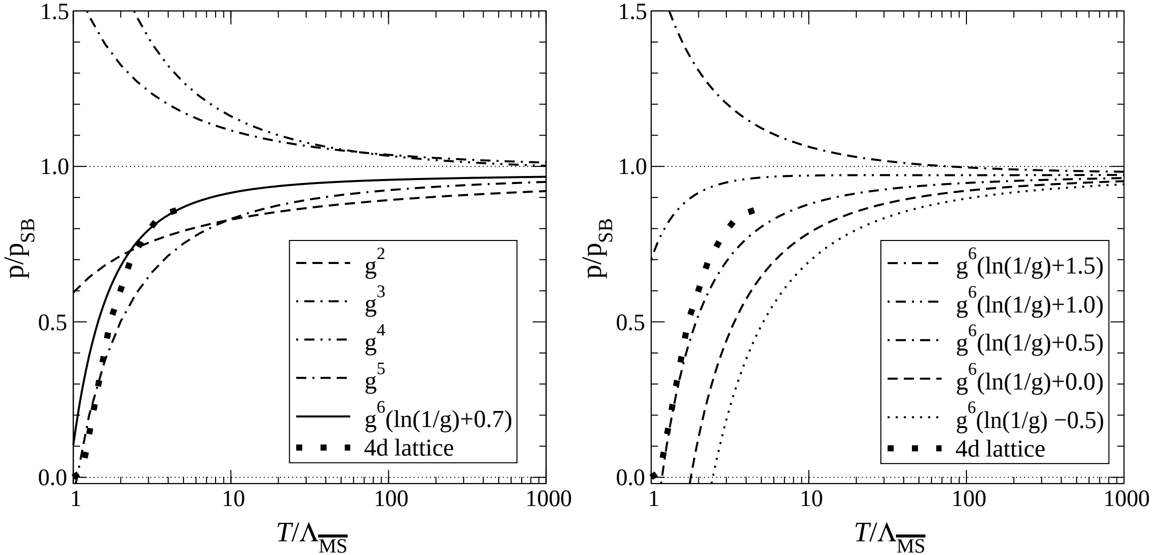
The perturbative expression for the pressure is now fully known:

$p/p_{\rm SB}$	$= 1$	Stefan-Boltzmann	
	$+g^2$	2-loop	(Shuryak 78)
	$+g^3$	resummed 2-loop	(Kapusta 79)
	$+g^4 \ln 1/g$	resummed 2-loop	(Toimela 83)
	$+g^4$	resum 3-loop	(Arnold, Zhai 94)
	$+g^5$	resum 3-loop	(Kastening, Zhai 95)
	$+g^6 \ln 1/g$	resum 4-loop	(Laine, Kajantie, K.R., Schroeder 02)
	$+g^6$	not computable in P.T!	(Linde 80)
	$+g^7 \ldots$	...	

$$p_{\rm SB} = \frac{\pi^2 T^4}{45}\left(8 - \frac{21 N_f}{4}\right)$$

However, P.T. alone does not provide the answer:

Laine, Kajantie, K.R, Schroeder 02



- Generally bad convergence
- Non-perturbative effects [parametrized here with $g^6 \times (const.)$] can be very significant!

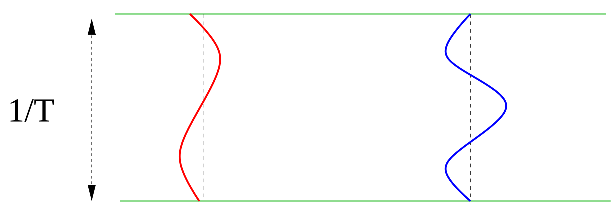
Why is g^6T^4 non-perturbative?

1) Finite T ensemble: euclidean metric with imaginary time extent $1/T$, with (anti)periodic b.c for bosons (fermions) $\Rightarrow$

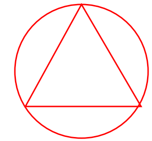
$$\frac{1}{p^2} \longrightarrow \frac{1}{\bar{p}^2+\omega_n^2}, \quad \omega_n = \begin{cases} 2n\pi T & n \in Z \quad \text{Bosons} \\ (2n+1)\pi T & n \in Z \quad \text{Fermions} \end{cases}$$

$$\int \frac{d^4p}{(2\pi)^4} \longrightarrow T\sum_n \int \frac{d^3p}{(2\pi)^3}$$

Thus, all $n \neq 0$ Bosonic modes and all Fermionic modes acquire a "mass" $\sim \pi T$. Clearly, only bosonic $n = 0$ modes are infrared sensitive.



2) Let us consider a vacuum gauge diagram at finite T, where we add a mass term m to keep track of the scale: Linde 80



$$N \text{ loops} \rightarrow \begin{cases} (N-1) & \text{4-vertices} \\ (2N-2) & \text{propagators} \end{cases}$$

$$\left[T\int d^3p\right]^N (g^2)^{N-1}\left[\frac{1}{q^2+m^2}\right]^{2N-2} = \# \times g^6 T^4 \left[\frac{g^2 T}{m}\right]^{N-4}$$

If $\boxed{m = g^2 T}$ ("magnetic" scale), all orders of the loop expansion contribute to pressure at g^6!

Perturbatively $m = 0$ for magnetic gauge field modes $\Rightarrow$ IR singularity at 4 or larger loop order

Note: for

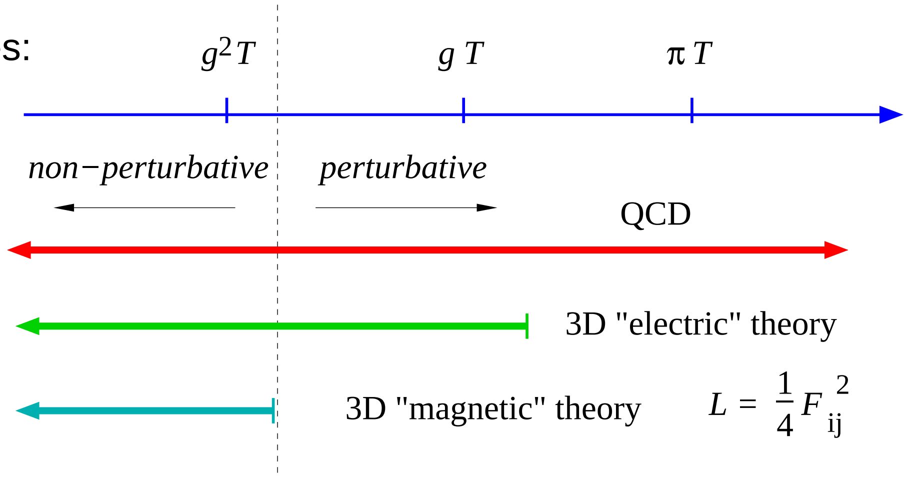
$m \sim \pi T$ (hard scales, $n \neq 0$)

$m \sim gT$ (Debye, electric scales)

the loop expansion is OK.

3d effective theory

Energy scales:



We obtain a hierarchy of effective theories

Braaten,Nieto 95

- Integrate over $\pi T \Rightarrow$ 3-dim. effective theory $\mathcal{L}_{\mathrm{E}}$ for gT, g^2T -modes
- Integrate over $gT \Rightarrow$ 3-dim. effective theory $\mathcal{L}_{\mathrm{M}}$ for g^2T -modes

(“integrate” = 2-loop optimized matching of theories)

Effective theory $\mathcal{L}_{\rm E}$ is 3d adjoint Higgs model:

$$\boxed{\mathcal{L}_{\rm E} = \frac{1}{2}\,{\rm Tr}\,F_{ij}^2 + {\rm Tr}[D_i, A_0]^2 + m_D^2\,{\rm Tr}\,A_0^2 + \lambda_A({\rm Tr}\,A_0^2)^2}$$

For $N_f = 0$ (pure gauge), the couplings are

$$\begin{aligned} g_3^2 &= g^2 T = \frac{8\pi^2}{11\ln(6.742T/\Lambda_{\overline{\rm MS}})}\,T \\ m_D^2 &\sim g^2T^2 \\ \lambda_A &\sim g^4 T \end{aligned}$$

$\mathcal{L}_{\rm E}$ can be analyzed using perturbation theory or lattice simulations. No problems in going to (almost) arbitrarily high T! (renormalization on the lattice can be done perturbatively)

$\mathcal{L}_{\rm M}$ is just 3d QCD: $\boxed{\mathcal{L}_{\rm M} = \frac{1}{2}\,{\rm Tr}\,F_{ij}^2}$

with coupling constant $g_3^2 \approx g^2T$.

Pressure:

pert. theory

$$\frac{p}{p_{\rm SB}} = 1 + g^2 + g^3 + g^4 \ln\frac{1}{g} + g^4 + g^5 + g^6 \ln\frac{1}{g} + g^6 + g^7 + \ldots$$

$\mathcal{L}_{\rm E}$

$\mathcal{L}_{\rm M}$

The relation between physical pressure and 3d free energy is (Braaten,Nieto)

$$\frac{p}{p_{\rm SB}} = 1 - \frac{5}{2}\frac{\lambda_A}{g_3^2} - \frac{45}{8\pi^2}\left(\frac{g_3^2}{T}\right)^3 (\mathcal{F}_E + \textbf{(known)})$$

where

$$\mathcal{F}_E = -\frac{1}{V g_3^6}\ln\int[dA]\exp[-\int d^3x\mathcal{L}_{\rm E}]$$

$\mathcal{F}_E$ can now be calculated

– perturbatively $\mapsto g^6 \ln\frac{1}{g}$

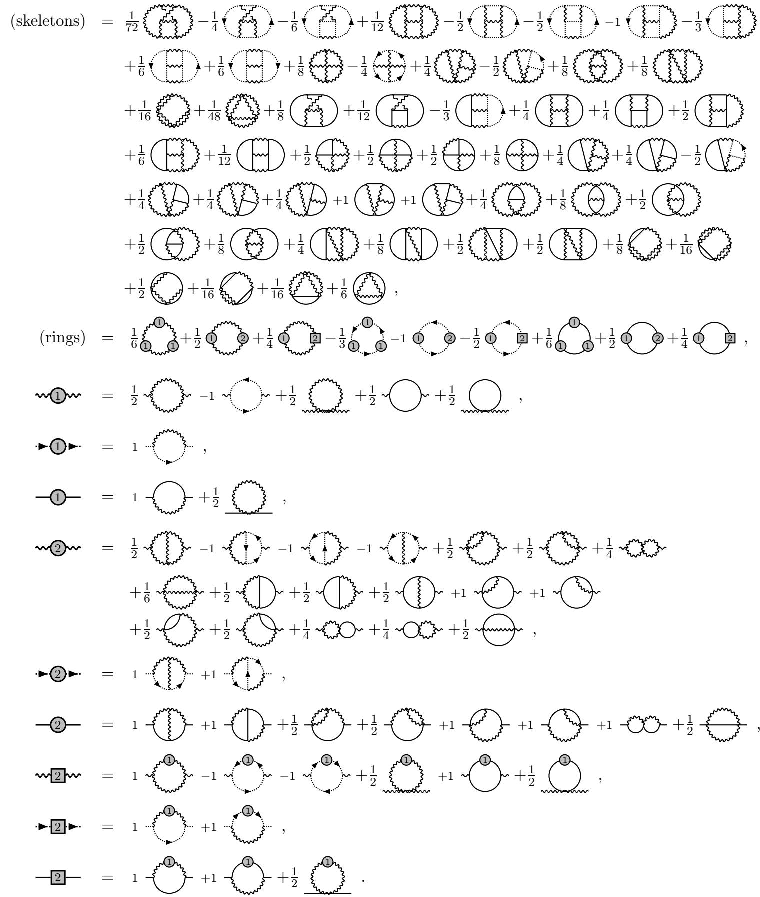
– non-perturbatively $\mapsto$ higher order

4-loop graphs for $g^6 \ln \frac{1}{g}$:

of diagrams $\sim$
3:6:47:490 at 1:2:3:4 loops

One 4-loop diag. with 6 3-point vertices and 9 propagators contains $\sim 24 \times 10^6$ terms, to be integrated over!
($2^9 \times (3 \times 2)^6$)

Computer algebra program is a *must* to sort out the integrals (FORM)



Non-perturbative contributions

The cleanest way to obtain the genuinely 3-dimensional g^6T^4 contribution is to use $\mathcal{L}_\mathrm{M}$, i.e. 3d QCD. g^6-term is directly related to the condensate $\langle F_{ij}^2 \rangle$. (Braaten, Nieto 95),(Karsch, Lütgemeier, Patkos, Rank 96)

This can be determined on the lattice. However, this requires 4-loop lattice$\leftrightarrow$continuum matching:

$$\begin{aligned}\langle F_{ij}^2 \rangle &= \frac{1}{a^3} + \frac{g^2T}{a^2} + \frac{(g^2T)^2}{a} \\ &+ (g^2T)^3(\ln\frac{1}{g^2Ta} + [\text{matching coeff.}] + [\text{non-pert. physics}])\end{aligned}$$

This has not yet been done (stochastic perturbation theory?)

Use $\mathcal{L}_E$ to calculate $\mathcal{F}_E$:

We cannot calculate the 3-dim. free energy $\mathcal{F}_E$ directly from lattice simulation. However, we can calculate *derivatives* of it:

$$\frac{d\mathcal{F}_E}{dy} = \frac{\partial \mathcal{F}_E}{\partial y} + \frac{\partial x}{\partial y}\frac{\partial \mathcal{F}_E}{\partial x} = \langle A_0^2 \rangle + C \langle A_0^4 \rangle$$

where $y \equiv m_D^2/g_3^4$, $x \equiv \lambda_A/g_3^2$ are dimensionless coupling constants. Condensates $\langle A_0^2 \rangle$ and $\langle A_0^4 \rangle$ are measurable from the lattice. $\mathcal{F}_E$ is obtained from integral

$$\mathcal{F}_E = \int dy \frac{d\mathcal{F}_E}{dy} + \text{const.}$$

The integration constant is precisely the “Linde term”! We can estimate it by fitting to 4-dim. data.

We drop the condensate $\langle A_0^4 \rangle$ here:

– its contribution is very small (parametrically and numerically)

– we don’t know all lattice counterterms for it

Strategy:

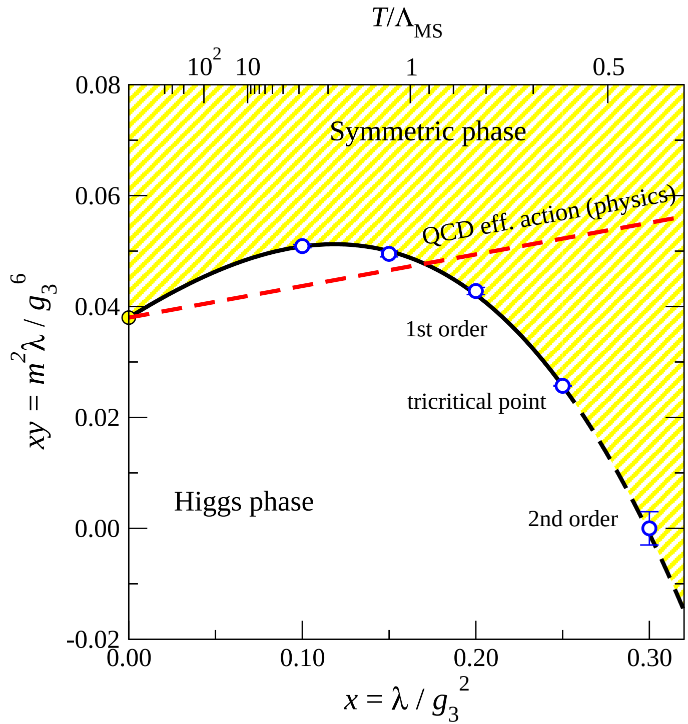
- Measure $\langle A_0^2 \rangle(a)$, subtract *lattice counterterms* (known) and extrapolate to continuum
- Subtract perturbative contributions
- Integrate, result gives $g^7 + g^8 + \ldots$ contributions to p
- add perturbative parts of p
- Tune g^6 term, until happy with the fit

– Bare $\langle A_0^2 \rangle_{\rm latt}$ has to be determined very precisely: 2 large subtractions! Accuracy to 4–5 decimal digits $\rightarrow$ large volumes (up to 200^3), large statistics.

– Reliable *continuum extrapolation:*
wide range of small lattice spacings $a = 0.37 \ldots 0.05\, g_3^{-2}$

Adjoint Higgs phase diagram

– $\exists$ phase transition, order parameter $\mathrm{Tr}\, A_0^3$.
– Effective theory must be in the symmetric phase. However, the line which corresponds to QCD lies in the broken phase!
– Solution: the symmetric phase is strongly ***metastable***. In practice is not a problem.



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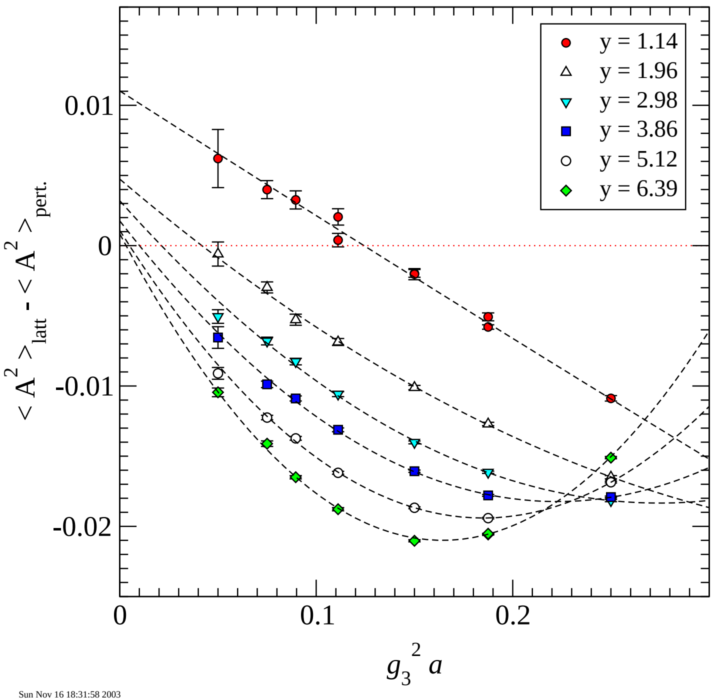
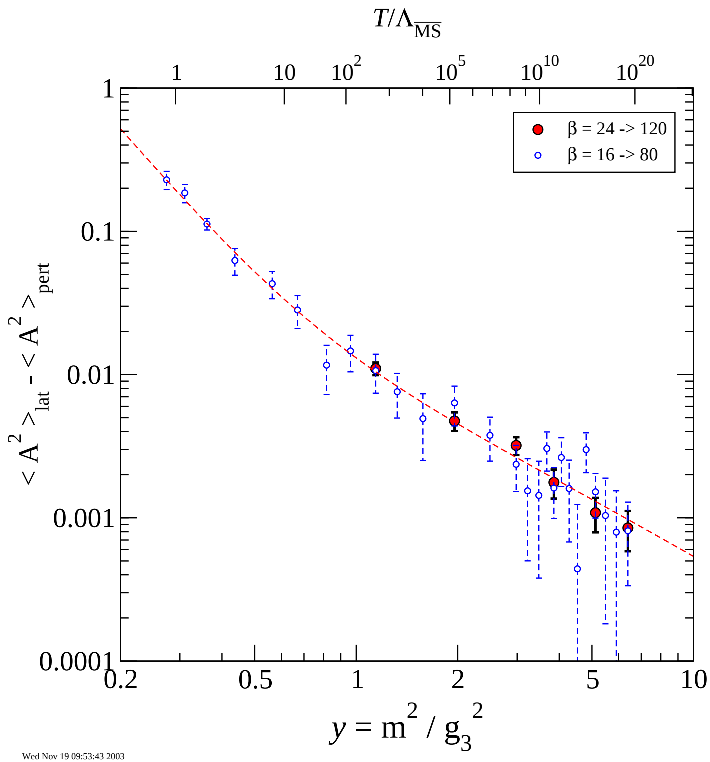
Continuum extrapolation

$y = m_D^2/g_3^4 \sim 1/g^2 = 1.14$ – 6.39, corresponding to $T \sim 100$ – $10^{20}\Lambda_{\overline{\mathrm{MS}}}$.



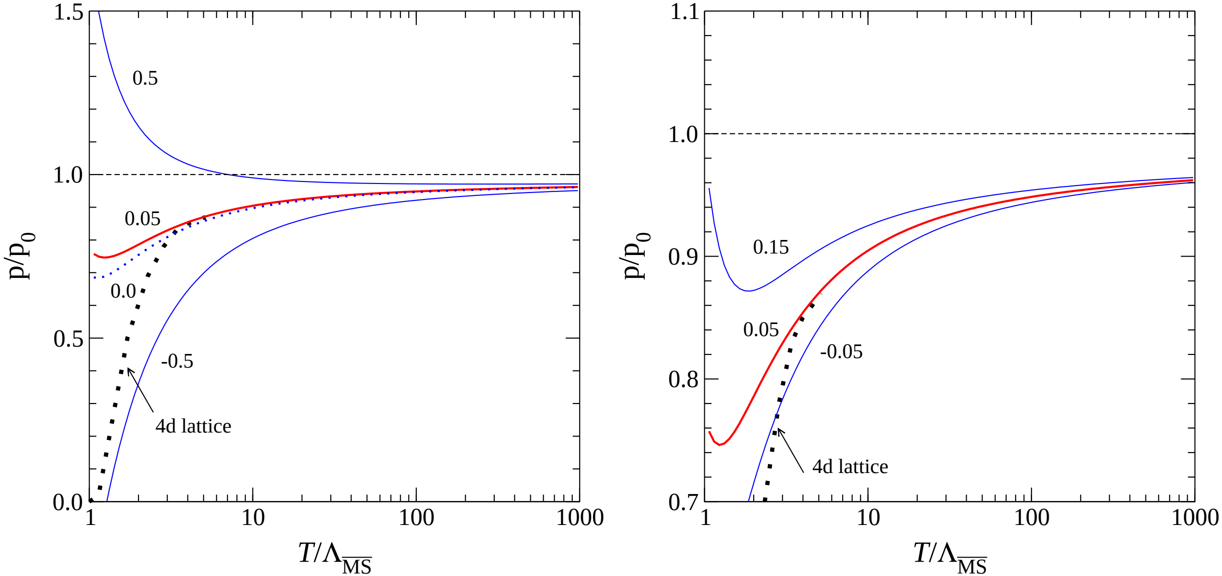
Non-perturbative part of the condensate

Integrate the condensate with an interpolating fit



Pressure

Tune g^6 coefficient to match 4d lattice:



Blue lines on the RHS plot: conservative error estimates for the g^6 coefficient

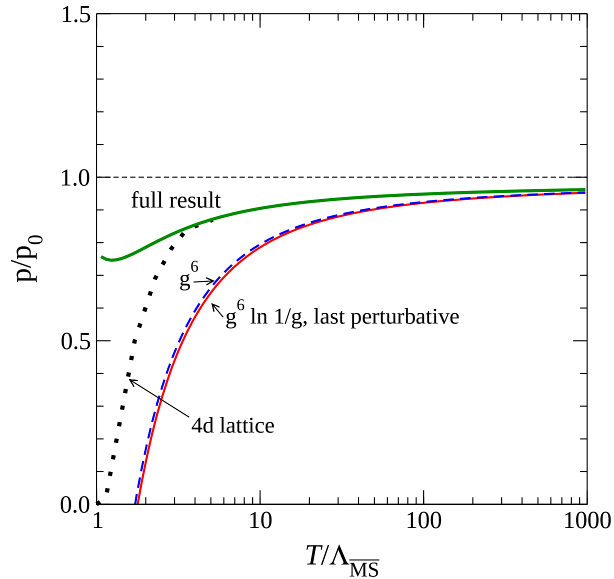
We can now write estimate the full $O(g^6 + g^7 + \ldots)$ contribution to $p/p_{\rm SB}$ as

$$0.0373 g^6 (\frac{1}{2} \ln \frac{1}{g} + C) - 0.015 g^7 + \ldots$$

where we estimate $C = -0.05 \ldots 0.15$. This is a very small value!

The coefficient of the g^7-term is not to be taken literally: it matches the data well, but the accuracy is not sufficient to disentangle the higher power coefficients separately.

Pressure, with full perturbative and non-perturbative contributions:



Conclusions

- Full expression for pressure in pure gauge QCD:

  $$p(T) = p_{\mathrm{pert.}}(T) + p_{\mathrm{non-pert}}(T)$$

- When perturbative expansion is organized as here, the coefficient of g^6T^4 -term is very small.
- Smallness of the g^6T^4 coefficient here does not imply that the free energy of $\mathcal{L}_M$, 3d QCD, is small! The coefficient is a sum of purely magnetic modes ($\mathcal{F}_M$) and other contributions, some perturbative, some not.
- The method used here relies on the accuracy of a) 4d lattice results, and b) $\mathcal{L}_E$ down to $T \sim 4T_c$.

  However, even allowing for quite conservative errors, the conclusions do not change qualitatively.
- Direct determination of the g^6T^4 contribution $\mathcal{L}_M$ is not restricted by these limitations.