QUASICONFORMAL FRAMES

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Abstract. We consider n-tuples of differential 1-forms in the Euclidean n-space that satisfy a quasiconformality condition and an asymptotic closedness condition. We show that renormalized sequences of such tuples have subsequences converging to differentials of quasiregular maps. We then use these maps to show that the tuples carry topological information.

1. Introduction

The study of quasiconformal or more general quasiregular mappings is a study of solutions $f = (f_1, \ldots, f_n)$ in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ to the system

$$(\det Df(x))^{-2/n}\, Df(x)^t Df(x) = G(x), \tag{1.1}$$

where G is a measurable matrix valued function in a domain Ω in $\mathbb{R}^n$, $n \geq 2$, satisfying, for some fixed $1 \leq K < \infty$,

$$\frac{1}{K}|\xi|^2 \leq \langle G(x)\xi, \xi\rangle \leq K|\xi|^2, \quad \xi \in \mathbb{R}^n,$$

almost everywhere in Ω. The general existence theory due to Morrey, Bojarski, and others lends the two dimensional theory a special flavor. In dimension $n \geq 3$, it is well known that the system (1.1) is overdetermined; moreover, there are no known *integrability conditions* that would guarantee the existence of solutions. See [12] for a thorough discussion on these matters.

In this paper, we approach the existence question from the point of view of approximative solutions, a method suggested by Sullivan [16], [17]. The matrix field G determines a conformal structure on Ω, and to a measurable Riemannian metric in this conformal class we can associate an orthonormal frame field ρ of positive orientation. This frame field, interpreted as a matrix field, satisfies the equation

$$(\det \rho(x))^{-2/n} \rho(x)^t \rho(x) = G(x). \tag{1.2}$$

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On the other hand, ρ can be naturally interpreted as a coframe of 1-forms. In what follows, we adopt this interpretation.

To describe our *a priori* assumptions on the frame field ρ, let us consider a quasiregular mapping $f\colon \Omega \to \mathbb{R}^n$. The 1-forms

$$\rho_i := df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j, \quad i = 1, \dots, n\ , \tag{1.3}$$

belong to $L^n_{\mathrm{loc}}(\Omega)$ and satisfy

$$|\rho|^n \leq K \star (\rho_1 \wedge \cdots \wedge \rho_n) \tag{QC}$$

almost everywhere in Ω, where by $|\rho|$ we mean the operator norm of the matrix $(\rho_{ij}) = \left(\frac{\partial f_i}{\partial x_j}\right)$ and by $\star(\rho_1 \wedge \cdots \wedge \rho_n)$ the density of $\rho_1 \wedge \cdots \wedge \rho_n$ with respect to the Euclidean volume element; $\star(\rho_1 \wedge \cdots \wedge \rho_n)$ coincides with the determinant of (ρ_{ij}) almost everywhere. The *frame* $\rho = (\rho_1, \dots, \rho_n)$ as in (1.3) is a pullback frame f^*dx of the standard Euclidean frame $dx = (dx_1, \dots, dx_n)$.

In general, we call an n-tuple $\rho = (\rho_1, \dots, \rho_n)$ of (Borel) measurable 1-forms in Ω a *measurable frame.* An obvious *integrability condition* for such a frame to be a pullback frame is that $d\rho = (d\rho_1, \dots, d\rho_n) = 0$ in the sense of distributions.

We do not want to assume such a strong condition. Instead, our objective is to study the asymptotic behavior of a frame at a point, and then find quasiregular mappings through a blow-up or renormalization procedure.

We now define conditions on a frame that will lead to germs of quasiregular mappings. Let $x_0 \in \mathbb{R}^n$. We call a (nonzero) measurable frame $\rho = (\rho_1, \dots, \rho_n)$ a *strong K-quasiconformal frame at x_0* if ρ is defined in a ball $B(x_0, r_0)$ about x_0 and satisfies integrability assumptions $\rho \in L^p$ for some $p > n$ and $d\rho \in L^q$ for some $q > n-1$, the *quasiconformality* condition (QC), the *strong doubling* condition

$$\not\|\,\rho\|_{p,B(x_0,r)} \leq C \not\|\,\rho\|_{n,B(x_0,r/2)} \quad \text{for every } 0 < r < r_0, \tag{SD}$$

and the *asymptotic closedness* condition

$$r\frac{\not\|\,d\rho\|_{q,B(x_0,r)}}{\not\|\,\rho\|_{n,B(x_0,r)}} \to 0 \quad \text{as } r \to 0. \tag{AC}$$

Here and in what follows we use the notation

$$\not\|\,u\|_{p,E} = \left(|E|^{-1} \int_E |u(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

Whereas we consider strong doubling to be a technical condition, the asymptotic closedness is a necessary condition for the renormalization process to produce mappings. The *strongness* of the frame refers to the high integrability assumption $q > n - 1$ of $d\rho$, see Section 2.

Although the imposed conditions seem restrictive, the differential $df = (df_1, \dots, df_n)$ of a quasiregular mapping $f\colon B(x_0, r_0) \to \mathbb{R}^n$ is a strong quasiconformal frame at x_0. Indeed, (QC) is the defining condition of quasiregularity; that (SD) is true follows from the higher integrability of the derivative of a quasiregular mapping and from the local doubling property [13], [1]. It is a rather deep fact that (SD) holds for $\rho = df$. Asymptotic closedness is obvious as $d\rho = d^2 f = 0$.

To describe the renormalization process, let ρ be a quasiconformal frame at x_0 and define a mapping f_ρ by

$$x \mapsto \int_{[x_0, x]} \rho.$$

It can be easily seen that the mapping f_ρ need not be quasiregular in any neighborhood of x_0. However, given a sequence $0 < r_i < r_0$ tending to zero, and maps f_i defined by

$$f_i(x) = \int_{[x_0, x_0 + r_i(x - x_0)]} \frac{\rho}{\|\rho\|_{n, B(x_0, r_i)}},$$

we may pass to a subsequence converging to a quasiregular tangent map.

For the statement of the first theorem, we say that a quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is *polynomial* (or *of polynomial type*) if $|f(x)| \to \infty$ as $|x| \to \infty$; cf. [9]. The degree $\deg(f)$ of such an f is the degree of its extension $\hat{f} : \mathbb{S}^n \to \mathbb{S}^n$.

Theorem A. *Suppose that ρ is a strong K-quasiconformal frame at x_0, and $r_i \searrow 0$. Then each f_i is continuous, and there exist a subsequence $\xi = (r_{i_j})$ and a polynomial K-quasiregular mapping $f_\xi\colon \mathbb{R}^n \to \mathbb{R}^n$ so that $f_{i_j} \to f_\xi$ locally uniformly.*

We denote the set of all mappings f_ξ obtained in Theorem A by $\mathcal{I}(x_0, \rho)$, and call this set the *infinitesimal space of ρ at x_0*. The infinitesimal space of a quasiregular mapping was introduced and thoroughly investigated by Gutlyanskii, Martio, Ryazanov, and Vuorinen [5]. In particular, they established Theorem A, as well as Theorem B below, in the case where ρ is a differential of a quasiregular mapping (a closed frame). Although the infinitesimal space of the frame may consist of several maps, it carries topological information on the frame as the following theorem presents.

Theorem B. *Suppose that ρ is a strong K-quasiconformal frame at x_0. Then all mappings in $\mathcal{I}(x_0, \rho)$ have the same positive degree.*

We define the *index of ρ at x_0*, denoted by $i(x_0, \rho)$, to be the common degree of the mappings in $\mathcal{I}(x_0, \rho)$. Our last main theorem shows that the index of a strong quasiconformal frame is stable in the following asymptotic sense. When ρ is a frame in $B(x_0, r_0)$, we define

$$\rho_r = \frac{\lambda_r^* \rho}{\|\rho\|_{n, B(x_0, r)}},$$

for every $0 < r < r_0$, where $\lambda_r(x) = x_0 + r(x - x_0)$.

Theorem C. *Suppose that ρ and $\tilde{\rho}$ are strong K-quasiconformal frames at x_0 and that*

$$\liminf_{r\to 0} \|\rho_r - \tilde{\rho}_r\|_{n,B(x_0,1)} < \varepsilon, \tag{1.4}$$

where $\varepsilon > 0$ depends only on n and the datas of ρ and $\tilde{\rho}$. Then $i(x_0, \rho) = i(x_0, \tilde{\rho})$.

Given Theorems A, B, and C, we obtain that, when ρ is a strong quasiconformal frame, infinitesimal solutions of Beltrami systems of type (1.2), are quasiregular mappings of the degree determined by the infinitesimal geometry of the frame. Moreover, the space of solutions is stable under natural perturbations.

The proof of Theorem A relies on three main ingredients. First, in Section 3, we consider a suitable extension of the smooth *Poincaré homotopy operator*. Then, in Section 4, we prove, with the aid of the results in [11], a *weak compactness* theorem for quasiconformal frames. This in turn implies a weak version of Theorem A for quasiconformal frames (Theorem 4.4), where a weak convergence in L^n of differentials instead of uniform convergence of mappings is concluded. In Section 5 we prove a *continuity estimate* for the maps f_i, which, together with the earlier results, finishes the proof of Theorem A. Theorems B and C are then proved by combining the uniform convergence property with methods from quasiregular mapping theory.

Although there is no formal connection, we would like to mention a recent work of Faraco and Zhong in the conformal case of the geometric rigidity problem of non-linear elasticity [2]; see also Friesecke, James, and Müller [3] and Rešetnjak [14]. It is tempting to consider the geometric rigidity problem as a question of finding such a mapping that its gradient field is a section of a given product fiber bundle, where the fiber is a given set of matrices. In such interpretation, the solutions to the Beltrami equation (1.1) correspond to finding mappings having gradient fields in a bundle, where the fibers are defined by the conformal structure G. For a more detailed discussion on the connections between non-linear elasticity and geometric function theory, we refer to [12, Section 1.12].

The notions of quasiconformal frames and Theorems A, B, and C generalize the theory of *Cartan-Whitney presentations* due to Sullivan [16], Heinonen and Sullivan [10], and Heinonen and Keith [7]. In the theory of Cartan-Whitney presentations, the bi-Lipschitz invariance and the *branch set* of the presentation lead to applications of the local theory to the parametrization and smoothability questions of Lipschitz manifolds. In the same spirit, we initiate in Sections 8 and 9 the study of the quasi-invariance and the branch set of quasiconformal frames. Section 9 ends with open questions in this direction.

Juha Heinonen passed away while this manuscript was being finished. P.P. and K.R. dedicate this work to his memory.

2. Notation

Given $n \geq 2$, $\ell \in \{0, \dots, n\}$, and a domain Ω in $\mathbb{R}^n$, we denote by $\Gamma(\bigwedge^\ell \Omega)$, $C^\infty(\bigwedge^\ell \Omega)$, and $C_0^\infty(\bigwedge^\ell \Omega)$ the spaces of *measurable ℓ-forms*, *smooth ℓ-forms*, and *smooth compactly supported ℓ-forms* on Ω, respectively.

The Euclidean metric on Ω induces an inner product $\langle \cdot, \cdot \rangle$ in the fibers of the exterior bundle $\bigwedge^\ell T\Omega$ of ℓ-covectors. This is uniquely determined by the requirement

$$\langle dx_I, dx_J \rangle = \begin{cases} 1, & I = J, \\ 0, & \text{otherwise}, \end{cases}$$

where $I = (i_1, \dots, i_\ell)$ and $J = (j_1, \dots, j_\ell)$ are ordered ℓ-tuples so that $1 \leq i_1 < \cdots < i_\ell \leq n$ and $1 \leq j_1 < \cdots < j_\ell \leq n$, and dx_I and dx_J are the ℓ-forms $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$ and $dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}$, respectively. We denote the associated norm and the *Hodge star operator* by $|\cdot|$ and by $\star$, respectively.

The L^p-norm of an ℓ-form $\omega \in \Gamma(\bigwedge^\ell \Omega)$ for $1 \leq p < \infty$ is defined by

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega|^p \, \mathrm{d}x \right)^{1/p},$$

and the L^∞-norm by

$$\|\omega\|_{\infty,\Omega} = \operatorname{esssup}_{x \in \Omega} |\omega(x)|.$$

The space of p-integrable ℓ-forms on Ω is denoted by $L^p(\bigwedge^\ell \Omega)$ and the corresponding local spaces by $L^p_{\mathrm{loc}}(\bigwedge^\ell \Omega)$. We write $\mathcal{H}^s$ for the Hausdorff s-measure and abbreviate $\mathrm{d}x = \mathrm{d}\mathcal{H}^n$ for the Lebesgue n-measure. We also use the notation

$$\fint \omega\|_{p,\Omega} = \left(|\Omega|^{-1} \int_\Omega |\omega(x)|^p \, \mathrm{d}x \right)^{1/p}$$

for $\omega \in L^p(\bigwedge^\ell \Omega)$.

The *weak exterior differential* of an ℓ-form $\omega \in L^1_{\mathrm{loc}}(\bigwedge^\ell \Omega)$ is the unique form $d\omega \in L^1_{\mathrm{loc}}(\bigwedge^{\ell+1} \Omega)$, if exists, that satisfies

$$\int_\Omega d\omega \wedge \varphi = (-1)^{\ell+1} \int_\Omega \omega \wedge d\varphi$$

for every $\varphi \in C_0^\infty(\bigwedge^{n-\ell-1} \Omega)$. We denote by $W_{p,q}(\bigwedge^\ell \Omega)$ *the (p,q)-partial Sobolev space* of ℓ-forms $\omega \in L^p(\bigwedge^\ell \Omega)$ having $d\omega \in L^q(\bigwedge^{\ell+1} \Omega)$.

We endow the fibers of the product bundle $(\bigwedge^\ell T\Omega)^n$ with the operator norm

$$|\xi| = \sup_{(v_1, \dots, v_\ell)} |(\xi_1(v_1, \dots, v_\ell)), \dots, \xi_n(v_1, \dots, v_\ell))|$$

where $\xi = (\xi_1, \dots, \xi_n)$ is an n-tuple of ℓ-covectors, and the supremum is taken over vectors $v_1, \dots, v_\ell \in T\Omega$ satisfying $\sum |v_i|^2 = 1$.

We call an n-tuple $\rho = (\rho_1, \dots, \rho_n)$ of (Borel) measurable 1-forms on Ω a *measurable frame*. We also say that a measurable frame is a $W_{p,q}$*-frame* if the forms ρ_i, $i = 1, \dots, n$, belong to $W_{p,q}$. We then denote

$$d\rho = (d\rho_1, \dots, d\rho_n).$$

A $W_{p,q}$-frame $\rho = (\rho_1, \dots, \rho_n)$ is a K*-quasiconformal frame* at x_0 if ρ is defined in a ball $B(x_0, r_0)$ about x_0, $p > n$, $q > n/2$, and if ρ satisfies (QC) and (AC) conditions together with the *doubling condition*

$$\text{(D)} \qquad \|\rho\|_{n,B(x_0,r)} \le C\|\rho\|_{n,B(x_0,r/2)} \quad \text{for every } 0 < r < r_0.$$

We denote by $B(x_0, r)$ the open n-ball centered at $x_0 \in \mathbb{R}^n$ of radius $r > 0$. We also write $B(r) = B(0, r)$ and $B^n = B(1)$, for short. The corresponding $(n-1)$-spheres are denoted by $S(x_0, r)$, $S(r)$, and S^{n-1}, respectively.

We let $C = C(a, b, \dots)$ denote a general constant that depends only on a, b, …, and whose value may vary from line to line.

3. The L^p-Poincaré homotopy operator

For two points $a, b \in \mathbb{R}^n$ we denote the (oriented) line segment from a to b by $[a, b]$. For a (pointwise defined) Borel measurable 1-form ω we set

$$\int_{[a,b]} \omega := \int_0^1 \omega(a + t(b-a); b-a)\, \mathrm{d}t$$

whenever the integral on the right exists.

The main result of this section is the following theorem on the existence and properties of the function $f_\rho \colon B \to \mathbb{R}$,

$$f_\rho(x) = \int_{[x_0,x]} \rho,$$

in a ball $B = B(x_0, r)$ for $\rho \in W_{p,q}(\bigwedge^1 B)$.

Theorem 3.1. *Suppose that $\rho \in W_{p,q}(\bigwedge^1 B)$, $B = B(x_0, r)$, for some $p > n$ and $q > n/2$. Then there exists $\alpha = \alpha(n, p, q) > 1$ so that $f_\rho \in W^{1,\alpha}(B)$. Moreover,*

$$\nparallel df_\rho\|_{\alpha,B} \le \nparallel \rho\|_{\alpha,B} + Cr \nparallel d\rho\|_{q,B},$$

where $C = C(n, \alpha, q) > 0$.

The proof of Theorem 3.1 is based on an extension of the Poincaré homotopy operator $\mathcal{K}_{x_0} \colon C^\infty(\bigwedge^\ell B) \to C^\infty(\bigwedge^{\ell-1} B)$,

$$\mathcal{K}_{x_0}\omega(x; v_1, \dots, v_{\ell-1}) = \int_0^1 t^{\ell-1}\omega(x_0 + t(x - x_0); x - x_0, v_1, \dots, v_{\ell-1})\, \mathrm{d}t,$$

where $\omega \in C^\infty(\bigwedge^\ell B)$ and $v_1, \dots, v_{\ell-1} \in \mathbb{R}^n$. Here and in what follows, we identify the fibers of $\bigwedge^1 TB$ with $\mathbb{R}^n$, as usual. For brevity, we write

$\mathcal{K}$ instead of $\mathcal{K}_0$ if B is a ball about the origin. For a smooth 1-form ρ on $B = B(x_0, r)$ we have $f_\rho = \mathcal{K}_{x_0}\rho$.

Remark 3.2. The operator $\mathcal{K}$ can be, for many purposes, replaced with the convoluted Poincaré homotopy operator $\mathcal{T}\colon L^p(\bigwedge^\ell B) \to L^p(\bigwedge^{\ell-1} B)$ of Iwaniec and Lutoborski [11]. See Section 6 for a discussion.

We note that given a Euclidean similarity map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\mathcal{K}_{A(x_0)}\rho = (A^{-1})^*\mathcal{K}_{x_0}(A^*\rho) \tag{3.1}$$

for every smooth ℓ-form ρ defined in a ball about $A(x_0)$. In particular, for every such 1-form,

$$f_\rho \circ A = \mathcal{K}_{x_0}A^*\rho.$$

Lemma 3.3. *Suppose that $\omega \in C^\infty(\bigwedge^\ell B)$ and $p > n/\ell$. Then*

$$\nparallel \mathcal{K}_{x_0}\omega\|_{\alpha,B} \leq Cr \nparallel \omega\|_{p,B}, \tag{3.2}$$

where $\alpha = p$ if $p \geq (n-1)/(\ell-1)$ and $1 < \alpha < p/(n - p(\ell-1))$ otherwise, and $C = C(n, \alpha, p, \ell) > 0$.

Proof. By (3.1) we may assume that $B = B(r)$. Given $p > n/\ell$, let α be as in the statement. First, we have

$$|\mathcal{K}\omega(x)|^\alpha \leq C\Big(\int_0^1 t^{\ell-1}|x||\omega(tx)|\,\mathrm{d}t\Big)^\alpha \leq C|x|^\alpha \int_0^1 t^{(\ell-1)\alpha}|\omega(tx)|^\alpha\,\mathrm{d}t$$

by Hölder's inequality. Next we integrate over $S^{n-1}(s)$, $0 < s < r$. By Fubini's theorem and change of variables $y = tx$,

$$\begin{aligned}\int_{S^{n-1}(s)} |\mathcal{K}\omega(x)|^\alpha\,\mathrm{d}\mathcal{H}^{n-1}(x) \quad &\leq \quad Cs^\alpha \int_0^1 \int_{S^{n-1}(s)} t^{(\ell-1)\alpha}|\omega(tx)|^\alpha\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ &= \quad Cs^\alpha \int_0^1 t^\beta \int_{S^{n-1}(ts)} |\omega(x)|^\alpha\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t,\end{aligned}$$

where $\beta = (\ell-1)\alpha + 1 - n$. Then, by change of variables $\eta = ts$, the last term equals

$$Cs^{\alpha-1-\beta} \int_0^s t^\beta \int_{S^{n-1}(t)} |\omega(x)|^\alpha\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t. \tag{3.3}$$

We split the rest of the proof to two cases. First, if $p \geq (n-1)/(\ell-1)$, then $\beta \geq 0$. We estimate the t-term and use Fubini's theorem to see that (3.3) is bounded from above by

$$Cs^{\alpha-1}\int_{B(s)} |\omega(x)|^\alpha\,\mathrm{d}x.$$

Thus,

$$\|\mathcal{K}\omega\|^\alpha_{\alpha,B(r)} \leq C\int_0^r s^{\alpha-1}\int_{B(s)} |\omega(x)|^\alpha\,\mathrm{d}x\,\mathrm{d}s \leq Cr^\alpha \|\omega\|^\alpha_{\alpha,B(r)},$$

which yields (3.2).

If $p < (n-1)/(\ell-1)$, then $\beta < 0$ and $\alpha - 1 - \beta > 0$. Thus (3.3) and Fubini's theorem yield

$$\begin{aligned}\|\mathcal{K}\omega\|^{\alpha}_{\alpha,B(r)} &\leq C\int_0^r s^{\alpha-1-\beta}\int_0^s t^{\beta}\int_{S^{n-1}(t)}|\omega(x)|^{\alpha}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t\,\mathrm{d}s \\ &\leq Cr^{\alpha-\beta}\int_{B(r)}|x|^{\beta}|\omega(x)|^{\alpha}\,\mathrm{d}x.\end{aligned}$$

We use Hölder's inequality

$$r^{\alpha-\beta}\int_{B(r)}|x|^{\beta}|\omega(x)|^{\alpha}\,\mathrm{d}x \leq r^{\alpha-\beta}\Big(\int_{B(r)}|x|^{\gamma}\,\mathrm{d}x\Big)^{(p-\alpha)/p}||\omega||^{\alpha}_{p,B(r)},$$

where $\gamma = \beta p/(p-\alpha)$. By our choice of α, $\gamma > -n$, and so the last term is bounded from above by

$$Cr^{\alpha+n-n\alpha/p}||\omega||^{\alpha}_{p,B(r)},$$

where $C = C(n,\alpha,p,\ell) > 0$. Thus (3.2) holds. The proof is complete. $\square$

By Lemma 3.3, and density of smooth forms in $L^p(\bigwedge^{\ell} B)$, the extension $\mathcal{K}_{x_0}\colon L^p(\bigwedge^{\ell} B) \to L^{\alpha}(\bigwedge^{\ell-1} B)$ is well-defined and (3.2) holds. Recall that in the smooth case $\mathcal{K}_{x_0}$ is a chain homotopy between identity and zero, that is,

$$\omega = \mathcal{K}_{x_0}d\omega + d\mathcal{K}_{x_0}\omega. \tag{3.4}$$

We next show that this identity remains valid under suitable Sobolev regularity assumptions.

Lemma 3.4. *Suppose that $\omega \in W_{p,q}(\bigwedge^{\ell} B)$ for some $p > n/\ell$ and $q > n/(\ell+1)$. Then (3.4) holds.*

Proof. We may assume that $B = B^n$. By density, there exists a sequence (ω_i) of smooth forms in B^n so that $\omega_i \to \omega$ in $L^p(\bigwedge^{\ell} B^n)$ and $d\omega_i \to d\omega$ in $L^q(\bigwedge^{\ell+1} B^n)$. We fix $\eta \in C_0^{\infty}(\bigwedge^{\ell} B^n)$. Then

$$\begin{aligned}\int_{B^n}\langle\omega - \mathcal{K}d\omega,\eta\rangle &= \int_{B^n}\langle\omega-\omega_i,\eta\rangle - \int_{B^n}\langle\mathcal{K}d\omega - \mathcal{K}d\omega_i,\eta\rangle \\ &\quad + \int_{B^n}\langle\omega_i - \mathcal{K}d\omega_i,\eta\rangle.\end{aligned}$$

The first term on the right hand side tends to zero as $i \to \infty$ by the L^p-convergence, and the second by Lemma 3.3 and our choice of q. For the last term we have

$$\int_{B^n}\langle\omega_i - \mathcal{K}d\omega_i,\eta\rangle = \int_{B^n}\langle d\mathcal{K}\omega_i,\eta\rangle = \int_{B^n}\langle\mathcal{K}\omega_i,d^*\eta\rangle \to \int_{B^n}\langle\mathcal{K}\omega,d^*\eta\rangle$$

by (3.4), Lemma 3.3, and our choice of p. The proof is complete. $\square$

Proof of Theorem 3.1. We may assume that $B = B(r)$. By Lemmas 3.3 and 3.4, $\mathcal{K}\rho$ is integrable, and

$$\begin{aligned} \nparallel d\mathcal{K}\rho\|_{\alpha,B} &= \nparallel \rho - \mathcal{K}d\rho\|_{\alpha,B} \leq \nparallel \rho\|_{\alpha,B} + \nparallel \mathcal{K}d\rho\|_{\alpha,B} \\ &\leq \nparallel \rho\|_{\alpha,B} + Cr \nparallel d\rho\|_{q,B}, \end{aligned}$$

where $C = C(n, \alpha, q) > 0$. Thus $\mathcal{K}\rho \in W^{1,\alpha}(B)$. On the other hand, $\mathcal{K}\rho = f_\rho$ almost everywhere in B by Fuglede's lemma [4, Theorem 3(f)]. The proof is complete. □

4. Weak compactness of quasiconformal frames

In this section we turn to the setting of quasiconformal frames. Recall that, given a frame $\rho = (\rho_1, \ldots, \rho_n)$ in a ball $B = B(x_0, r_0)$, and $0 < r < r_0$, we denote

$$\rho_r = \frac{\lambda_r^* \rho}{\|\rho\|_{n,B(x_0,r)}}, \quad \lambda(x) = x_0 + r(x - x_0),$$

and $f_r = \mathcal{K}_{x_0}\rho_r$. Given a sequence (r_i), we denote $\rho_i = \rho_{r_i}$ and $f_i = \mathcal{K}_{x_0}\rho_i$ if there is no ambiguity.

Theorem 4.1. *Suppose that ρ is a $W_{p,q}$-frame in $B(x_0, r)$ for $p > n$ and $q > n/2$ satisfying* (QC) *with some $K \geq 1$ and* (AC)*. Let $r_i \searrow 0$. Then there exist a subsequence $\xi = (r_{i_j})$ and a K-quasiregular mapping $f_\xi \colon B(x_0, 1) \to \mathbb{R}^n$ such that*

$$\rho_{i_j}|B(x_0, 1) \to df_\xi$$

weakly in L^n.

The following two-dimensional example shows that the limit map f_ξ need not be quasiregular if (AC) is relaxed.

Example 4.2. We define $\rho = (\rho_1, \rho_2)$ by $\rho_1 = x_1 dx_1 + x_2 dx_2$, $\rho_2 = -x_2 dx_1 + x_1 dx_2$. Then ρ satisfies (QC) with $K = 1$. Moreover, $|d\rho| \equiv 1$ and

$$r \frac{\nparallel d\rho\|_{q,B(r)}}{\nparallel \rho\|_{2,B(r)}} = \sqrt{2}$$

for every $r > 0$ and $q > 1$. However, $f_\xi(x) - f_\xi(0) = (2^{1/2}|x|^2/\pi^{1/2}, 0)$ for every limit map f_ξ in Theorem 4.1. These mappings are not quasiregular or even discrete.

We do not know if it is possible for the map f_ξ in Theorem 4.1 to be a constant map. This is in fact equivalent to the question whether Condition (D) follows from the other conditions in the definition of a quasiconformal frame. That (D) is true for closed frames i.e. differentials of quasiregular mappings depends on a deep theorem of Rešetnjak: quasiregular mappings are branched covering maps, cf. [15]. We would need an extension of Rešetnjak's theorem to our current setting in order to be able to show that our maps f_ξ are always nontrivial.

The following example shows that constant maps can arise if the assumption (AC) is replaced by

$$\liminf_{r\to 0} r\frac{\nparallel d\rho\|_{q,B(x_0,r)}}{\nparallel \rho\|_{n,B(x_0,r)}} = 0. \tag{4.1}$$

Example 4.3. In this example, we consider 1-frames on $\mathbb{R}^2$ as complex valued 1-forms on $\mathbb{C}$ for notational brevity. This convention allows us to consider the pull-back frame $(f_k^*dx_1, f_k^*dx_2)$, where $f_k\colon \mathbb{C}\to\mathbb{C}$ is the mapping $f_k(z) = z^k$, as a 1-form dz^k. As $dz^k = kz^{k-1}dz$, we have that $|dz^k| = k|z|^{k-1}$.

Fix $1 < q < 2$. Let (r_k) be a decreasing sequence so that $r_{k+1}/r_k \to 0$ as $k\to\infty$. We assume for simplicity that $r_0 = 1$ and $3r_{k+1} < r_k$ for every $k \geq 1$. Let $\{\alpha_k\}$ be a smooth partition of unity, $k \geq 1$, so that $\operatorname{supp}\alpha_k \subset B(r_k)\setminus \bar{B}(r_{k+1}/2)$, $\alpha_k \equiv 1$ on $B(r_k/2)\setminus\bar{B}(r_{k+1})$, and $\|d\alpha_k\|_\infty \leq 4/r_{k+1}$.

We set ρ to be the 1-frame

$$\rho = \sum_{k=1}^\infty \alpha_k dz^k = \left(\sum_{k=1}^\infty \alpha_k k z^{k-1}\right) dz$$

on B^2. Clearly, ρ satisfies (QC) with $K = 1$.

To see that ρ satisfies (4.1), we first observe that we have the pointwise estimate $|d\rho| \leq Ckr_k^{k-3}$ on $B(r_k)\setminus\bar{B}(r_k/2)$ for every k, where $C > 0$ is a constant. Since $d\rho = 0$ in the complement of these annuli, we have that

$$\int_{B(r_k/2)} |d\rho|^q \;=\; \int_{B(r_{k+1})} |d\rho|^q \leq Ck^q r_{k+1}^{(k-2)q} r_{k+1}^2$$

and

$$\nparallel d\rho\|_{q,B(r_k/2)} \leq Ckr_k^{k-2}\left(\frac{r_{k+1}}{r_k}\right)^{2/q}.$$

Furthermore, we have the estimate

$$\nparallel \rho\|_{2,B(r_k/2)} \geq Ckr_k^{k-1}$$

for every k, where $C > 0$ is a constant. Hence along the sequence $(r_k/2)$ we have

$$(r_k/2)\frac{\nparallel d\rho\|_{q,B(r_k/2)}}{\nparallel \rho\|_{2,B(r_k/2)}} \leq C\left(\frac{r_{k+1}}{r_k}\right)^{2/q} \to 0$$

as $k\to\infty$. Thus ρ satisfies (4.1). However, ρ does not satisfy (AC), which can be seen as follows. By the pointwise estimate

$$|d\rho| \geq C(k-1)|z|^{k-2}|d\alpha_{k-1}\wedge dz| = C(k-1)|z|^{k-2}|d\alpha_{k-1}|$$

on $B(r_k)\setminus\bar{B}(r_k/2)$, we have

$$\begin{aligned}\int_{B(r_k)} |d\rho|^q &\geq C(k-1)^q r_k^{(k-2)q}\int_{B(r_k)\setminus\bar{B}(r_k/2)} |d\alpha_{k-1}|^q \\ &\geq C(k-1)^q r_k^{(k-2)q}\operatorname{cap}_q(B(r_k),\bar{B}(r_k/2)) \\ &= C(k-1)^q r_k^{(k-2)q} r_k^{2-q},\end{aligned}$$

where cap_q is the (variational) q-capacity, see [15, II 10]. Since $|\rho| \le 2k|z|^{k-2}$ on $B(r_k)$, we have

$$r_k \frac{\not\| d\rho\|_{q,B(r_k)}}{\not\| \rho\|_{2,B(r_k)}} \quad \ge \quad r_k \frac{C(k-1)r_k^{k-2+(2-q)/q-2/q}}{Ckr_k^{k-2}} \ge C > 0$$

for every k. Thus ρ does not satisfy (AC).

Let (t_k) be the sequence $t_k = r_k/3$ and set $\rho_k = \rho_{t_k}$. Let $\xi = (t_{i_j})$ be a subsequence of (t_k). We show that if there exists a mapping $f \in W^{1,2}(B^2;\mathbb{R}^2)$ so that $\rho_{i_j}|B^2 \to df$ weakly in L^2 then f is a constant map.

Let $0 < R < R' < 1$. Since

$$\int_{B(Rr_k/3)} |\rho|^2 \le \int_{B(Rr_k/3)} |dz^k|^2 \le Ck^2(Rr_k/3)^{2k}$$

and

$$\begin{aligned} \int_{B(r_k/3)\setminus B(R'r_k/3)} |\rho|^2 \quad &\ge \quad \int_{B(r_k/3)\setminus B(R'r_k/3)} |dz^k|^2 \\ &\ge \quad Ck^2(R'r_k/3)^{2(k-1)}((r_k/3)^2 - (R'r_k/3)^2), \end{aligned}$$

where $C > 0$ is a constant, we have that

$$\|\rho_k\|_{2,B(R)} = \frac{\|\rho\|_{2,B(Rr_k/3)}}{\|\rho\|_{2,B(r_k/3)}} \le \frac{\|\rho\|_{2,B(Rr_k/3)}}{\|\rho\|_{2,B(r_k/3)\setminus B(R'r_k/3)}} \to 0 \quad \text{as } k \to \infty. \tag{4.2}$$

Thus, if there exists $f \in W^{1,2}(B^2;\mathbb{R}^2)$ so that $\rho_{i_j} \to df$ weakly in L^2 then f is a constant mapping. By (4.2), ρ does not satisfy (D).

As mentioned above, the doubling condition guarantees the nontriviality of the tangent maps.

Theorem 4.4. *Suppose that ρ is a K-quasiconformal frame at x_0, and $r_i \searrow 0$. Then there exist a subsequence $\xi = (r_{i_j})$ and a polynomial K-quasiregular mapping $f_\xi \colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\rho_{i_j} \to df_\xi$$

locally weakly in L^n.

For the proofs of Theorems 4.1 and 4.4, we first note that

$$\|\rho_r\|_{p,B(x_0,t)} = C(n,p)t^{n/p} \frac{\not\| \rho\|_{p,B(x_0,tr)}}{\not\| \rho\|_{n,B(x_0,r)}} \tag{4.3}$$

and

$$\|d\rho_r\|_{q,B(x_0,t)} = C(n,q)t^{n/q} r \frac{\not\| d\rho\|_{q,B(x_0,tr)}}{\not\| \rho\|_{n,B(x_0,r)}}, \tag{4.4}$$

where $C(n,s) = |B^n|^{\frac{1}{s}-\frac{1}{n}}$. In particular, $\|\rho_r\|_{n,B(x_0,1)} = 1$.

The proof of Theorem 4.1 is based on the weak compactness of mappings f_{ρ_i} in $W^{1,\alpha}$ and on the weak L^n-compactness of frames satisfying (QC).

Proposition 4.5. *Let $K \geq 1$ and (ρ_i) be a sequence of frames, bounded in $W_{n,q}(\bigwedge^1 B^n)$ for some $q > n/2$, converging weakly in L^n to a $W_{n,q}$-frame $\rho = (\rho_1, \ldots, \rho_n)$. If the frames ρ_i satisfy Condition* (QC) *with K, then ρ satisfies* (QC) *with K.*

Proof. By the convexity of $t \mapsto t^n$,

$$|\rho_i|^n - |\rho|^n \geq n|\rho|^{n-1}\left(|\rho_i| - |\rho|\right)$$

for every i. Thus

$$\int_{B^n} \eta|\rho|^n \quad \leq \quad \liminf_{i\to\infty} \left(\int_{B^n} \eta|\rho_i|^n - n\int_{B^n} \eta|\rho|^{n-1}(|\rho_i| - |\rho|) \right)$$

for every $\eta \in C_0^\infty(B^n)$. Since $\eta|\rho|^{n-1} \in L^{n/(n-1)}(B^n)$ and $|\rho_i| \to |\rho|$ weakly in L^n, we have

$$\int_{B^n} \eta|\rho|^n \leq \liminf_{i\to\infty} \int_{B^n} \eta|\rho_i|^n \leq \liminf_{i\to\infty} K \int_{B^n} \eta \star ((\rho_i)_1 \wedge \cdots \wedge (\rho_i)_n)\,. \tag{4.5}$$

Since $q > n/2$, we have by compensated compactness [11, Theorem 5.1] that

$$\lim_{i\to\infty} \int_{B^n} \eta \star ((\rho_i)_1 \wedge \cdots \wedge (\rho_i)_n) = \int_{B^n} \eta \star (\rho_1 \wedge \cdots \wedge \rho_n)\,. \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$|\rho|^n \leq K \star (\rho_1 \wedge \cdots \wedge \rho_n)$$

almost everywhere in B^n. The proof is complete. □

Proof of Theorem 4.1. We may assume that the frames ρ_i are defined in B^n and $x_0 = 0$. By (4.3), (4.4), and (AC), (ρ_i) is bounded in $W_{n,q}$. Hence there exist a subsequence, also denoted by (r_i), and a frame ρ in B^n so that $\rho_i \to \rho$ weakly in L^n. By Proposition 4.5, ρ satisfies (QC) with K.

Now we consider the sequence $f_i = \mathcal{K}\rho_i$. Combining Theorem 3.1, (4.3), (4.4), and (AC) yields

$$\nparallel df_i\|_{\alpha,B^n} \leq \nparallel \rho_i\|_{\alpha,B^n} + C \nparallel d\rho_i\|_{q,B^n} \leq C$$

for some $\alpha > 1$. This together with the Sobolev-Poincaré inequality shows that $(\tilde{f_i}) = (f_i - (f_i)_B)$ is a bounded sequence in $W^{1,\alpha}(B^n;\mathbb{R}^n)$, where $(f_i)_B$ is the mean value of f_i in B^n;

$$(f_i)_B = \left(|B^n|^{-1} \int_{B^n} (f_i)_1, \ldots, |B^n|^{-1} \int_{B^n} (f_i)_n \right).$$

Thus, by the weak compactness of Sobolev spaces [8, Theorem 1.31], there exist a subsequence, also denoted by (r_i), and $f \in W^{1,\alpha}(B^n;\mathbb{R}^n)$, so that $d\tilde{f_i} \to df$ weakly in L^α. To show that f is K-quasiregular, it is now sufficient to show that $\rho = df$.

Given $\eta \in C_0^\infty(B^n)$, we have

$$\left|\int_{B^n} \eta(df - \rho)\right| \leq \left|\int_{B^n} \eta(df - df_i)\right| + \left|\int_{B^n} \eta(df_i - \rho_i)\right| + \left|\int_{B^n} \eta(\rho_i - \rho)\right|,$$

where the integrals are considered as vector valued. The first and the third terms on the right hand side tend to zero as i tends to infinity by the weak convergence in L^α and L^n, respectively. To see that the second term tends to zero, we use Lemmas 3.4 and 3.3 together with (4.3), (4.4), and (AC) to obtain

$$\begin{aligned} \|df_i - \rho_i\|_{\alpha,B^n} &= \|d\mathcal{K}\rho_i - \rho_i\|_{\alpha,B^n} = \|\mathcal{K}d\rho_i\|_{\alpha,B^n} \\ &\leq C\|d\rho_i\|_{q,B^n} = Cr_i\,\frac{\nparallel d\rho\|_{q,B(r_i)}}{\nparallel \rho\|_{n,B(r_i)}} \to 0, \end{aligned}$$

as $i \to \infty$. Hence

$$\int_{B^n} \eta(df - \rho) = 0$$

for every $\eta \in C_0^\infty(B^n)$. If follows that $df = \rho$ almost everywhere in B^n. The proof is complete. □

Proof of Theorem 4.4. We assume that $x_0 = 0$, and fix $k \in \mathbb{N}$. Then by (D), and (4.3), the sequence $(\rho_i|B(k))$ is bounded in L^n. Now we can use the proof of Theorem 4.1 to show that there exist a subsequence (r_{i_j}) and a K-quasiregular mapping $f_k\colon B(k) \to \mathbb{R}^n$ so that

$$\rho_{i_j}|B(k) \to df_k$$

weakly in L^n. By taking a diagonal subsequence (ρ^k) we then deduce that there exists a K-quasiregular mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\rho^k \to df$$

locally weakly in L^n.

To prove that f is polynomial, it suffices to show that $0 < ||J_f||_{1,B(2r)} \leq C||J_f||_{1,B(r)}$ for every $0 < r < \infty$, see the proof of [9, 1.5]. We fix η_1 and η_2 in $C_0^\infty(\mathbb{R}^n)$ so that $0 \leq \eta_1 \leq 1$ and $0 \leq \eta_2 \leq 1$, $\eta_1 = 1$ on $B(r/2)$ and 0 in $\mathbb{R}^n \setminus B(r)$, and $\eta_2 = 1$ on $B(2r)$ and 0 in $\mathbb{R}^n \setminus B(4r)$. Then, by [11, Theorem 5.1] and (D),

$$\begin{aligned} ||J_f||_{1,B(2r)} &\leq \int_{\mathbb{R}^n} \eta_2 J_f = \lim_{k\to\infty} \int_{\mathbb{R}^n} \eta_2 \star (\rho_1^k \wedge \cdots \wedge \rho_n^k) \\ &\leq \lim_{k\to\infty} \int_{B(4r)} \star(\rho_1^k \wedge \cdots \wedge \rho_n^k) \leq C \lim_{k\to\infty} \int_{B(r/2)} \star(\rho_1^k \wedge \cdots \wedge \rho_n^k) \\ &\leq C \lim_{k\to\infty} \int_{\mathbb{R}^n} \eta_1 \star (\rho_1^k \wedge \cdots \wedge \rho_n^k) = C \int_{\mathbb{R}^n} \eta_1 J_f \leq C||J_f||_{1,B(r)}. \end{aligned}$$

This proves the doubling property. On the other hand, if we consider $r = 2$ and recall that $||\rho^k||_{n,B^n} = 1$, the calculation above, combined with (QC), shows that $||J_f||_{1,B(2)} > 0$. The proof is complete. □

5. Continuity and the proof of Theorem A

In this section we first return to the setting of Section 3, and prove a Hölder continuity estimate for the function f_ρ. Together with Theorem 4.4 this leads to the proof of Theorem A. The following theorem corresponds to the Hölder continuity of $W^{1,p}$-functions with $p > n$.

Theorem 5.1. *Suppose that $\rho \in W_{p,q}(\bigwedge^1 B)$ for some $p > n$ and $q > n-1$, where $B = B(x_0, r_0)$. Then f_ρ has a continuous representative; for every x and $y \in B$,*

$$|f_\rho(x) - f_\rho(y)| \leq C \left(|x-y|^{1-n/p} \|\rho\|_{p,B} + |x-y|^{1-(n-1)/q} \|d\rho\|_{q,B} \right), \tag{5.1}$$

where $C = C(n,p,q) > 0$.

It is well-known that Sobolev functions in $W^{1,n}$ need not be continuous. Respectively, when $n \geq 3$, Theorem 5.1 is not true for 1-forms in $W_{p,n-1}$ as the following example shows.

Example 5.2. Recall that a singleton $\{a\} \subset S^{n-1}$ has zero $(n-1)$-capacity in S^{n-1} when $n \geq 3$; there exists $u \in W^{1,n-1}(S^{n-1})$ such that $u \in L^p(S^{n-1})$ for every $1 \leq p < \infty$ and $|u(x)| \to \infty$ as $x \to e_n$ or $x \to -e_n$. We define a 1-form ρ in B^n by $\rho(x) = u(x/|x|)dx_n$. Then $|\rho(x)| = |u(x/|x|)|$ and $|d\rho(x)| \leq C|x|^{-1}|\nabla u(x/|x|)|$ almost everywhere. Hence $\rho \in W_{p,n-1}(\bigwedge^1 B^n)$ for every $1 \leq p < \infty$. However, f_ρ is not continuous because all its representatives are unbounded around the x_n-axis.

Proof of Theorem 5.1. We may assume that $B = B^n$. By density and Fuglede's lemma [4, Theorem 3(f)], we may assume that ρ is smooth. Indeed, given a sequence of smooth forms ρ_i converging to ρ in $L^p(\bigwedge^1 B^n)$, Fuglede's lemma implies that

$$f_i(x) - f_i(y) = \int_{[y,x]} \rho_i \to \int_{[y,x]} \rho = f_\rho(x) - f_\rho(y)$$

for every x and y in $B^n \setminus E$, where $|E| = 0$. Here $f_i = f_{\rho_i}$, and from now on we denote $f = f_\rho$.

Now fix $x \in B^n \setminus \{0\}$ and $0 < r < |x|/4$. We may assume that $x = |x|e_n$. We denote $B' = B(x,r) \cap B^n$, and $f_{B'}$ is the average of f in B'. We will give an estimate for $|f(x) - f_{B'}|$. By Stokes' theorem,

$$\begin{aligned} |f(x) - f_{B'}| &\leq |B'|^{-1} \int_{B'} |f(x) - f(y)| \,\mathrm{d}y \\ &= |B'|^{-1} \int_{B'} \left| \int_{[0,x]} \rho - \int_{[0,y]} \rho + \int_{[x,y]} \rho - \int_{[x,y]} \rho \right| \,\mathrm{d}y \\ &\leq |B'|^{-1} \int_{B'} \left(\int_{[0,x,y]} |d\rho| \,\mathrm{d}\mathcal{H}^2 \right) \mathrm{d}y + |B'|^{-1} \int_{B'} \left(\int_{[y,x]} |\rho| \right) \mathrm{d}y. \end{aligned} \tag{5.2}$$

The last term can be estimated by Fubini's theorem, the change of variables, and Hölder's inequality in a standard way:

$$\begin{aligned}\int_{B'}\left(\int_{[x,y]}|\rho|\right)\,\mathrm{d}y &= \int_0^1\int_{B'}|\rho|(x+t(y-x))|y-x|\,\mathrm{d}y\,\mathrm{d}t\\ &= \int_0^r\int_0^s\int_{S(x,u)\cap B^n}|\rho|(z)\left(\frac{s}{u}\right)^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}(z)\,\mathrm{d}u\,\mathrm{d}s\\ &\le C(n,p)r^n r^{1-n/p}\|\rho\|_{p,B^n}.\end{aligned}$$

For later use we notice that the same estimate gives

$$|f_{B(x,|x|/4)\cap B^n}|\le C|x|^{-n}\int_{B(5|x|/4)\cap B^n}|f(y)|\,\mathrm{d}y\le C|x|^{1-n/p}||\rho||_{p,B^n}. \tag{5.3}$$

To estimate the $d\rho$-term in (5.2) we need additional notation. We will use the $(n-1)$-balls

$$P^{n-1}(s,t)=(\mathbb{R}^{n-1}\times\{se_n\})\cap B(se_n,t),$$

and their boundaries

$$T^{n-2}(s,t)=(\mathbb{R}^{n-1}\times\{se_n\})\cap S(se_n,t).$$

Set also $R=\max_{y\in\overline{B'}}|y|$ and $x'=Re_n$. We extend $|d\rho|$ outside B^n as the zero function. Then

$$\int_{B'}\left(\int_{[0,x,y]}|d\rho|\,\mathrm{d}\mathcal{H}^2\right)\,\mathrm{d}y\le C(n)r^2\int_{T^{n-2}(R,2r)}\left(\int_{[0,x',z]}|d\rho|\,\mathrm{d}\mathcal{H}^2\right)\,\mathrm{d}\mathcal{H}^{n-2}(z).$$

By the change of variables and Hölder's inequality, we have

$$\begin{aligned}&\int_{T^{n-2}(R,2r)}\left(\int_{[0,x',z]}|d\rho|\,\mathrm{d}\mathcal{H}^2\right)\,\mathrm{d}\mathcal{H}^{n-2}(z)\\ &= \int_{T^{n-2}(R,2r)}\int_0^R\int_0^{2rt/R}|d\rho|\left(t\frac{x'}{|x'|}+s\frac{z-x'}{|z-x'|}\right)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-2}(z)\\ &= \int_0^R\int_0^{2rt/R}\int_{T^{n-2}(t,s)}|d\rho|(y)\left(\frac{2r}{s}\right)^{n-2}\,\mathrm{d}\mathcal{H}^{n-2}(y)\,\mathrm{d}s\,\mathrm{d}t\\ &= (2r)^{n-2}\int_0^R\int_{P^{n-1}(t,2rt/R)}\frac{|d\rho|(y)}{|y-te_n|^{n-2}}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t\\ &\le C(n)r^{n-2}\int_0^R\left(\int_{P^{n-1}(t,2rt/R)}|d\rho|^q(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)^{1/q}(tr)^{1-(n-1)/q}\,\mathrm{d}t\\ &\le C(n,q)r^{n-2}r^{1-(n-1)/q}\|d\rho\|_{q,B^n}.\end{aligned}$$

By combining these estimates, we finally have

$$|f(x)-f_{B'}|\le C(n,p,q)\left(r^{1-n/p}\|\rho\|_{p,B^n}+r^{1-(n-1)/q}\|d\rho\|_{q,B^n}\right). \tag{5.4}$$

Now we will use the above estimates for x and y in $B^n \setminus \{0\}$. First, if $|x-y| < |x|/8$,

$$B'(y,|x-y|) \subset B'(x,2|x-y|) \subset B'(x,|x|/4),$$

where B' is the intersection of the corresponding ball with B^n. Then

$$\begin{aligned}
|f(x)-f(y)| &\le |f(x)-f_{B'(y,|x-y|)}| + |f(y)-f_{B'(y,|x-y|)}| \\
&\le C|x-y|^{-n}\int_{B'(x,2|x-y|)} |f(x)-f(z)|\,\mathrm{d}z \\
&+ |f(y)-f_{B'(y,|x-y|)}|.
\end{aligned}$$

Notice that (5.4) remains true with $|f(x)-f_{B'(x,2|x-y|)}|$ replaced by

$$|x-y|^{-n}\int_{B'(x,2|x-y|)} |f(x)-f(z)|\,\mathrm{d}z.$$

Thus (5.1) follows from (5.4). We are left with the case $|x-y| \ge |x|/8$. We may assume that $|x| \ge |y|$. Now

$$\begin{aligned}
|f(x)-f(y)| &\le |f(x)-f_{B'(x,|x|/4)}| + |f_{B'(x,|x|/4)}| + |f_{B'(y,|y|/4)}| \\
&+ |f(y)-f_{B'(y,|y|/4)}|.
\end{aligned}$$

This combined with (5.3), (5.4) and our assumption $|x| < 8|x-y|$ yields (5.1). The proof is complete. □

Now we are ready to complete the proof of Theorem A.

Proof of Theorem A. Suppose that $r_i \searrow 0$, and fix $R \ge 1$. By Theorem 5.1, we may assume that the mappings f_i are continuous. Furthermore, by (5.1), (4.3), (4.4), (SD), and (AC), we have the continuity estimate

$$\begin{aligned}
|f_i(x)-f_i(y)| &\le C\left(|x-y|^{1-n/p}\|\rho_i\|_{p,B(R)} + |x-y|^{1-(n-1)/q}\|d\rho_i\|_{q,B(R)}\right) \\
&\le C\left(|x-y|^{1-n/p} + |x-y|^{1-(n-1)/q}\right)
\end{aligned}$$

for every x and y in $B(R)$, with C not depending on i. We conclude that $(f_i|B(R))$ is equicontinuous. Hence, by the Arzela-Ascoli theorem, there exists a subsequence (f_{i_j}) converging locally uniformly to $f : \mathbb{R}^n \to \mathbb{R}^n$. By the proof of Theorem 4.4, (f_{i_j}) has a further subsequence, also denoted by (f_{i_j}), converging to a polynomial K-quasiregular mapping f_ξ locally weakly in $W^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^n)$ for some $\alpha > 1$. Clearly $f_\xi = f$. The proof is complete. □

6. Proof of Theorem B

In this section we prove the following result, and show how it implies Theorem B.

Theorem 6.1. *Suppose that ρ is a strong quasiconformal frame at x_0. Then there exists a radius $\varepsilon > 0$ so that*

$$B(x_0,\varepsilon) \cap f_\rho^{-1}(0) = \{x_0\}.$$

In our proof of Theorem 6.1, on which Theorems B and C depend, it is important that the map f_ρ is constructed by using the operator $\mathcal{K}$, that is, by integrating ρ over radial segments. One can also construct maps from a given frame by using other operators, such as the averaged homotopy operator $\mathcal{T}$ considered in Section 9 below. The advantage of $\mathcal{T}$ over $\mathcal{K}$ is that it has nicer analytic behavior, see [11]. However, the averaged operator $\mathcal{T}$ does not naturally commute with the rescaling induced by λ_r whereas operator $\mathcal{K}$ does. Indeed, using operator $\mathcal{K}$, the rescalings can be viewed as blow-ups of a single map $f_\rho = \mathcal{K}\rho$. If follows that the rescalings are easier to analyze when using the operator $\mathcal{K}$. We do not know whether Theorems B and C hold if the operator $\mathcal{T}$ is used to produce the infinitesimal space.

To prove Theorem 6.1 we use the following corollary of Theorem A. As the proof of the corollary follows directly from the compactness of the rotation group, we omit the details.

Corollary 6.2. *Suppose that ρ is a strong K-quasiconformal frame at x_0, and $r_i \searrow 0$. Define $g_i = f_i \circ h_i$, where $f_i = f_{\rho_{r_i}}$ and h_i is a rotation about the origin. Then there exist a subsequence $\xi = (g_{i_j})$, and a polynomial K-quasiregular mapping $f_\xi : \mathbb{R}^n \to \mathbb{R}^n$ so that $g_{i_j} \to f_\xi$ locally uniformly.*

We will also use the following local distortion estimate for quasiregular mappings, see [15, II 4.3] for details. Suppose that $f : B(x_0, r_0) \to \mathbb{R}^n$ is a non-constant K-quasiregular mapping. Then there exist a constant $H' \geq 1$ and a radius $s_0 > 0$ so that

$$H_f(x_0, s) \leq H' \tag{6.1}$$

for every $0 < s < s_0$. Here

$$H_f(x,s) = \frac{\max_{y \in S(x,s)} |f(y) - f(x)|}{\min_{y \in S(x,s)} |f(y) - f(x)|}.$$

In what follows, we assume that ρ is a strong quasiconformal frame at x_0. Without loss of generality, $x_0 = 0$. We will use some basic properties of the local topological degree $\mu(y, f, U)$, cf. [15, I 4].

Lemma 6.3. *There exist a sequence (s_i), decreasing to 0, and a constant $H \geq 1$, so that*

$$H_{f_\rho}(0,s) \leq H \quad \text{and} \quad \mu(0, f_\rho, B(s)) \geq 1 \tag{6.2}$$

for every $s_i/5 \leq s \leq 5s_i$. In particular, $S(s) \cap f_\rho^{-1}(0) = \emptyset$ for every such s.

Proof. By Theorem A, there exist a non-constant quasiregular mapping $f \colon \mathbb{R}^n \to \mathbb{R}^n$ and a sequence (r_i) so that $f_i \to f$ locally uniformly. Moreover, by (6.1), $H_f(0,t) \leq H'$ for $0 < t \leq t_0$. Thus, by the uniform convergence, there exists $i_0 \in \mathbb{N}$ so that $H_{f_i}(0,t) \leq 2H'$ and $\mu(0, f_i, B(t)) = \mu(0, f, B(t)) \geq 1$ for every $i \geq i_0$ and $t_0/5 \leq t \leq 5t_0$. The claim follows since $H_{f_\rho}(0,s) = H_{f_i}(0, s/r_i)$ and $\mu(0, f_\rho, B(s)) = \mu(0, f_i, B(s/r_i))$. $\square$

Lemma 6.4. *There exists* $\varepsilon_0 > 0$ *so that for every* $x \in f_\rho^{-1}(0) \cap B(\varepsilon_0) \setminus \{0\}$ *there exists* $0 < s_x < |x|/10$ *with the following properties:*

(1) $f_\rho^{-1}(0) \cap \bar{B}(x, 5s_x) \setminus B(x, s_x/5) = \emptyset$ *and*
(2) $\mu(0, f_\rho, B(x, s_x/5)) \geq 1$.

Proof. We argue by contradiction: suppose that for every $\epsilon > 0$ there exists $x \in f_\rho^{-1}(0) \cap B(\varepsilon) \setminus \{0\}$ such that the required s_x does not exist. Then there exists a sequence (x_i) of such points so that $x_i \to 0$ as $i \to \infty$. Define $g_i = f_{|x_i|} \circ h_i$, where h_i is a rotation about the origin so that $h_i(e_1) = x_i/|x_i|$. Then $g_i(e_1) = 0$ for every i. By Corollary 6.2, there exists a subsequence of (g_i) that converges locally uniformly to a non-constant quasiregular mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$. By redefining the sequence (x_i), we may assume that the whole sequence (g_i) converges to f. Then $f(e_1) = 0$, and by (6.1), there exists $H' \geq 1$ so that $H_f(e_1, t) \leq H'$ for every $0 < t < 5t_0$. We may assume that $t_0 < 1/10$. Then, by the uniform convergence,

$$H_{f_\rho}(x_i, t|x_i|) = H_{g_i}(e_1, t) \leq 2H',$$

and

$$\mu(0, f_\rho, B(x_i, t|x_i|)) = \mu(0, g_i, B(e_1, t)) \geq 1$$

for every $i \geq i_0$ and $t_0/5 \leq t \leq 5t_0$. By the definition of H_{f_ρ}, $f_\rho^{-1}(0) \cap \bar{B}(x_i, 5|x_i|t_0) \setminus B(x_i, |x_i|t_0/5) = \emptyset$. This contradicts our choice of the points x_i. The proof is complete. $\square$

Proof of Theorem 6.1. We choose a decreasing sequence (r_i) as in Lemma 6.3, so that $r_1 < \varepsilon_0$, where ε_0 is as in Lemma 6.4. We denote $A_i = \bar{B}(r_1) \setminus B(r_i)$ and $\tilde{A}_i = \bar{B}(5r_1) \setminus B(r_i/5)$ for all $i \geq 1$. We also set $F = f_\rho^{-1}(0) \cap B(\varepsilon_0) \setminus \{0\}$. Let $\mathcal{B}_i'$ be the covering of $A_i \cap F$ by the balls $B(x, s_x) \subset \tilde{A}_i$, where s_x is as in Lemma 6.4. By the $5r$-covering lemma, there exists a finite or countable subfamily $\mathcal{B}_i = \{B(x_j, s_j)\}$ of $\mathcal{B}_i'$ covering $A_i \cap F$ so that the balls $B(x_j, s_j/5)$ are pairwise disjoint. Since $f_\rho(S(5r_1)) \cap \{0\} = \emptyset$, the topological degree $\mu(0, f_\rho, B(5r_1))$ is well-defined (and thus finite). By the additivity of the topological degree, cf. [15, I 4.4], and Lemma 6.3,

$$\begin{aligned}
\mu(0, f_\rho, B(5r_1)) &= \mu(0, f_\rho, B(r_i/5)) + \mu(0, f_\rho, \tilde{A}_i) \\
&\geq 1 + \sum_j \mu(0, f_\rho, B(x_j, s_j/5)).
\end{aligned}$$

By Lemma 6.4, $\mu(0, f_\rho, B(x_j, s_j/5)) \geq 1$ for every j. We conclude that

$$1 + \operatorname{card} \mathcal{B}_i \leq \mu(0, f_\rho, B(5r_1)) < \infty,$$

i.e. there are at most finitely many balls $B(x_j, s_j/5)$ in the collection, with an upper bound not depending on i. Since $|s_j| < |x_j|/10$ for each j, this shows that $B(r_i) \cap F = \emptyset$ for i large enough. The proof is complete. $\square$

Proof of Theorem B. We assume that $x_0 = 0$. From Theorem 6.1 and Lemma 6.3 it follows that

$$m = \mu(0, f_\rho, B(s)) = \mu(0, f_\rho, B(\varepsilon)) \geq 1$$

for every $s < \varepsilon$, where ε is the radius in Theorem 6.1. Let f_ξ be a mapping in the infinitesimal space $\mathcal{I}(0, \rho)$ and $\xi = (r_i)$. We claim that f_ξ has degree m. Since f_ξ is a discrete map, $f_\xi^{-1}(0) \cap S(0,t) = \emptyset$ for almost every $t > 0$. Then, by uniform convergence, choosing i to be large enough yields

$$\mu(0, f_\xi, B(t)) = \mu(0, f_\rho, B(tr_i)) = m. \tag{6.3}$$

Since the degree is additive, (6.3) proves our claim, and, consequently, Theorem B. □

Remark 6.5. The proof of Theorem B also shows that $f^{-1}(0) = \{x_0\}$ for every $f \in \mathcal{I}(x_0, \rho)$.

Remark 6.6. By [9] and the proof of Theorem A, the degrees of the mappings in $\mathcal{I}(x_0, \rho)$ are bounded from above by a constant only depending on n and the data of ρ.

7. Proof of Theorem C

In this section we prove Theorem C on the stability of the index $i(x_0, \cdot)$ of strong quasiconformal frames. Here by the data of a quasiconformal frame we mean the constants K and C in (QC) and (SD), respectively.

Notice that if the limes inferior in (1.4) equals 0, then there exists f that is a limit map for both ρ and ρ'. Thus in that case $i(x_0, \rho) = i(x_0, \rho')$ automatically follows from Theorem B.

We first recall a familiar continuity estimate for quasiregular mappings, cf. [12, 7.7.1]. Suppose that $f : B^n \to \mathbb{R}^n$ is a K-quasiregular mapping. There exist $C_0 \geq 1$ and $\alpha > 0$, only depending on n and K, so that

$$|f(x) - f(y)|^n \leq C_0 |x - y|^{n\alpha} ||J_f||_{1,B^n} \tag{7.1}$$

for every $x, y \in \bar{B}(1/2)$.

Our second auxiliary result is a distortion estimate for a special class of quasiregular mappings. In the proof we use the path family method. Since it does not appear elsewhere in this paper, we do not give all details and definitions; they can be found in [15].

Lemma 7.1. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial K-quasiregular mapping with $f^{-1}(0) = \{0\}$. Then*

$$\min_{y \in S^{n-1}} |f(y)| \geq A,$$

where $A > 0$ depends only on n, K, $\deg(f)$, *and* $\|J_f\|_{1,B(1/2)}$.

Proof. By the Caccioppoli inequality, cf. [15, VI (3.15)], and the openness of f,

$$\|J_f\|_{1,B(1/2)} \leq C \max_{y \in S^{n-1}} |f(y)|^n,$$

where $C > 0$ only depends on n and K. Hence it suffices to show that $H_f(0,1)$ is bounded from above by a constant depending only on n, K, and $\deg(f)$; recall the definition of $H_f(x,s)$ from Section 6.

Fix a point $a \in S^{n-1}$ so that $|f(a)| = \min_{y\in S^{n-1}} |f(y)|$, and consider the a-component U of $f^{-1}(B(|f(a)|))$. Since f is a polynomial map, U is relatively compact. Thus U is a normal domain of f, and $f(\bar{U}) = \bar{B}(|f(a)|)$, see [15, I 4.7]. From our assumption $f^{-1}(0) = \{0\}$ it then follows that $0 \in U$.

Now fix $b \in S^{n-1}$ so that $|f(b)| = \max_{y\in S^{n-1}} |f(y)|$, and consider the maximal f-lifting γ' of γ starting at b, where $\gamma : [1,\infty) \to \mathbb{R}^n$, $\gamma(t) = tf(b)$ (for information on path lifting, see [15, II 3]). Then $|\gamma'|$ joins b and ∞. Denote by Γ the family of all paths joining $|\gamma'|$ and U in $\mathbb{R}^n$. Then standard path family estimates and the K_O-inequality (see [15, II 2.4]) give

$$c_n \le M_n\Gamma \le K \deg(f) M_n f\Gamma \le C_1(\log H_f(0,1))^{1-n},$$

where M_n is the n-modulus of path families, and C_1 only depends on n, K and $\deg(f)$. The proof is complete. $\square$

Proof of Theorem C. We assume $x_0 = 0$, and fix $\varepsilon > 0$, to be determined later. We choose a sequence (r_i) decreasing to 0 so that $||\rho_i - \rho_i'||_{n,B^n} < \varepsilon$ for each i. This can be done by (1.4). Without loss of generality, we may assume, by Theorem A that (f_{ρ_i}) and $(f_{\rho_i'})$ converge locally uniformly to polynomial quasiregular mappings f and g, respectively. By Remark 6.5, $f^{-1}(0) = g^{-1}(0) = \{0\}$.

As in the proof of Theorem 4.4, we see that

$$\min\{\|J_f\|_{1,B(1/2)}, \|J_g\|_{1,B(1/2)}\} \ge C^{-1},$$
$$\max\{||J_f||_{1,B(2)}, ||J_g||_{1,B(2)}\} \le C$$

and

$$||Df - Dg||_{n,B^n} < \varepsilon, \tag{7.2}$$

where $C \ge 1$ only depends on the datas of ρ and ρ'. In particular, (7.1) implies that there exists $\alpha = \alpha(n,K) > 0$ so that

$$\max\{|f(x) - f(y)|, |g(x) - g(y)|\} \le C|x - y|^\alpha \tag{7.3}$$

for every x and y in $\bar{B}^n$.

By Theorem B, it suffices to show that $\deg(\hat{f}) = \deg(\hat{g})$, where $\hat{f}$ and $\hat{g}$ are the extensions of f and g to mappings $\mathbb{S}^n \to \mathbb{S}^n$, respectively. Since $f^{-1}(0) = \{0\}$ and $g^{-1}(0) = \{0\}$, the extensions $\hat{f}$ and $\hat{g}$ have the same degree if the restrictions of $f/|f|$ and $g/|g|$ to S^{n-1} are homotopic as mappings $S^{n-1} \to S^{n-1}$. Thus it suffices to show that

$$\frac{f(a)}{|f(a)|} \ne -\frac{g(a)}{|g(a)|}$$

for every $a \in S^{n-1}$.

Let $a \in S^{n-1}$. By Lemma 7.1 and Remark 6.6,

$$\min\{|f(a)|, |g(a)|\} \ge A,$$

where $A > 0$ only depends on the datas of ρ and ρ'. Hence it suffices to show that

$$|f(a) - g(a)| < 2A.$$

We fix $\delta > 0$, to be determined later, and denote by T_δ the spherical cap $B(a,\delta) \cap S^{n-1}$. Also, we denote $h = f - g$. Then, by applying (7.3) twice and the triangle inequality, we obtain

$$\begin{aligned} |h(a)| &\le C\delta^\alpha + C\delta^{1-n} \int_{T_\delta} |h(x)|\, d\mathcal{H}^{n-1}(x) \\ &\le C\delta^\alpha + C\delta^{1-n} \int_{T_\delta} |h(x) - h(\delta x)|\, d\mathcal{H}^{n-1}(x) \\ &\le C\delta^\alpha + C\delta^{1-n} \int_{T_\delta} \int_\delta^1 |Dh(tx)|\, dt\, d\mathcal{H}^{n-1}(x). \end{aligned}$$

By the change of variables, Hölder's inequality and (7.2), the last integral is controlled by

$$\begin{aligned} \int_{B^n \setminus \bar{B}(\delta)} \frac{|Dh(y)|}{|y|^{n-1}}\, dy &\le \|Dh\|_{n,B^n} \Big(\int_{B^n \setminus \bar{B}(\delta)} |y|^{-n}\, dy \Big)^{(n-1)/n} \\ &\le C\varepsilon \big(\log \delta^{-1} \big)^{(n-1)/n}. \end{aligned}$$

Thus

$$|f(a) - g(a)| \le C\delta^\alpha + C\varepsilon\delta^{1-n} \big(\log \delta^{-1} \big)^{(n-1)/n},$$

where $C > 0$ only depends on n and the data. Now we can choose δ so that $C\delta^\alpha = A/2$, and then ε so that $C\varepsilon\delta^{1-n}\big(\log\delta^{-1}\big)^{(n-1)/n} = A/2$. The proof is complete. $\square$

8. Quasi-invariance

In this section we show that quasiconformal frames are preserved under pullbacks by quasiregular mappings.

Theorem 8.1. *Suppose that ρ is a K_0-quasiconformal frame at $f(x_0)$, where f is a non-constant K_1-quasiregular mapping. Then $f^*\rho$ is a K_0K_1-quasiconformal frame at x_0.*

Proof. We may assume that $x_0 = f(x_0) = 0$. Also, we assume that ρ is a K_0-quasiconformal frame at 0, satisfying $\rho \in L^{p_0}$ for some $p_0 > n$, and $d\rho \in L^{q_0}$ and (AC) for some $q_0 > n/2$. We denote, for $r > 0$, $L(r) = \max_{x \in S(r)} |f(x)|$ and $l(r) = \min_{x \in S(r)} |f(x)|$. Then there exist $A \ge 1$ and $r' > 0$ so that

$$L(r) \le Al(r/2) \tag{8.1}$$

for every $0 < r < r'$, see the proof of [15, II 4.3].

We will also use the reverse Hölder inequality of quasiregular mappings: there exist $C > 0$ and $\tau > 1$ so that

$$⨏\!\!\| J_f \|_{\tau,B(r)} \le C\, ⨏\!\!\| J_f \|_{1,B(r)} \tag{8.2}$$

when r is small enough, see [13].

We first show that $f^*\rho \in L^p(\bigwedge^1 B(r_0))$ for some $n < p < p_0$ and $r_0 > 0$. By the quasiregularity of f and Hölder's inequality,

$$\begin{aligned}\int_{B(r)} |f^*\rho(x)|^p \,\mathrm{d}x &\leq C\int_{B(r)} J_f(x)^{p/n}|\rho(f(x))|^p \,\mathrm{d}x \\ &\leq C\Big(\int_{B(r)} J_f(x)^t \,\mathrm{d}x\Big)^{(p_0-p)/p_0} \\ &\quad\times\Big(\int_{B(r)} J_f(x)|\rho(f(x))|^{p_0} \,\mathrm{d}x\Big)^{p/p_0},\end{aligned}$$

where

$$t = \frac{(p_0-n)p}{(p_0-p)n}.$$

By the p_0-integrability of ρ, and the change of variables, the last term is finite for r small enough. On the other hand, we can choose $n < p < p_0$ so that $t \leq \tau$, and apply (8.2) to show that also the J_f-term is finite when r is small.

To prove condition (QC) we recall that quasiregular mappings preserve sets of zero n-measure. Hence condition (QC) applied to ρ, and the quasiregularity of f, give

$$\begin{aligned}|f^*\rho(x)|^n &\leq |Df(x)|^n|\rho(f(x))|^n \leq K_1 J_f(x) K_0 \star(\rho_1\wedge\cdots\wedge\rho_n)(f(x)) \\ &= K_0K_1\star((f^*\rho)_1\wedge\cdots\wedge(f^*\rho)_n)(x)\end{aligned}$$

almost everywhere. Similarly, to prove condition (D) we fix a small $r > 0$ and calculate

$$\int_{B(r)} |f^*\rho(x)|^n \,\mathrm{d}x \leq K_1\int_{B(r)} J_f(x)|\rho(f(x))|^n \,\mathrm{d}x \leq C\int_{B(L(r))} |\rho(x)|^n \,\mathrm{d}x.$$

By (8.1), and condition (D) applied to ρ, the last term is bounded by

$$\begin{aligned}C\int_{B(l(r/2))} |\rho(x)|^n \,\mathrm{d}x &\leq C\int_{B(r/2)} J_f(x)|\rho(f(x))|^n \,\mathrm{d}x \\ &\leq C\int_{B(r/2)} |f^*\rho(x)|^n \,\mathrm{d}x.\end{aligned}$$

Here we used the inclusion $B(l(r/2)) \subset f(B(r/2))$ which is valid by the openness of f. Condition (D) follows.

Now we turn to the proof of (AC). We note that the identity $df^*\rho = f^*d\rho$ is valid under our assumptions. We fix $n/2 < q < q_0$ to be determined later.

First, by the quasiregularity of f and Hölder's inequality,

$$
\begin{aligned}
\not\| df^*\rho\|_{q,B(r)} &\leq C\Big(r^{-n}\int_{B(r)} J_f(x)^s J_f(x)^{q/q_0}|d\rho(f(x))|^q \,\mathrm{d}x\Big)^{1/q} \\
&\leq C\Big(r^{-n}\int_{B(r)} J_f(x)^{sq_0/(q_0-q)}\,\mathrm{d}x\Big)^{(q_0-q)/qq_0} \qquad (8.3)\\
&\quad\times\Big(r^{-n}\int_{B(r)} J_f(x)|d\rho(f(x))|^{q_0}\,\mathrm{d}x\Big)^{1/q_0},
\end{aligned}
$$

where $s = 2q/n - q/q_0$. We can choose $q > n/2$ so that

$$\frac{sq_0}{q_0-q} \leq \tau,$$

which allows us to apply (8.2) to obtain

$$
\begin{aligned}
\Big(r^{-n}\int_{B(r)} J_f(x)^{sq_0/(q_0-q)}\,\mathrm{d}x\Big)^{(q_0-q)/qq_0} &\leq C\Big(r^{-n}\int_{B(r)} J_f(x)\,\mathrm{d}x\Big)^{s/q} \\
&\leq C\Big(\frac{L(r)}{r}\Big)^{ns/q}.
\end{aligned}
$$

For the last term of (8.3), the change of variables, and (AC) applied to ρ give

$$
\begin{aligned}
\Big(r^{-n}\int_{B(r)} J_f(x)|d\rho(f(x))|^{q_0}\,\mathrm{d}x\Big)^{1/q_0} &\leq C\Big(\frac{L(r)}{r}\Big)^{n/q_0}\not\| d\rho\|_{q_0,B(L(r))} \\
&\leq \frac{\varepsilon(r)}{L(r)}\Big(\frac{L(r)}{r}\Big)^{n/q_0}\not\| \rho\|_{n,B(L(r))},
\end{aligned}
$$

where $\varepsilon(r)\to 0$ as $r\to 0$. Condition (D) applied to ρ, and (8.1) then yield

$$\not\|\rho\|_{n,B(L(r))} \leq C\not\|\rho\|_{n,B(l(r))} \leq C\frac{r}{L(r)}\not\| f^*\rho\|_{n,B(r)}$$

as above. Combining the estimates gives (AC) at x_0 for $f^*\rho$. The proof is complete. □

9. Quasiconformality of frames in a domain

In the previous sections we have studied the properties of quasiconformal frames at a point. In this section we show that if the asymptotic closedness condition (AC) of a frame is replaced by a stronger uniform condition, the strong doubling condition can be weakened to a corresponding doubling condition. This leads us to the notion of *(strong) quasiconformal frames in a domain*. At the end of this section we define the branch set of a strong quasiconformal frame and discuss open problems concerning the properties of these frames.

Let $\rho = (\rho_1,\ldots,\rho_n)$ be a $W_{n,q}$-frame in a domain Ω for some $q > n/2$. We say that ρ is *locally doubling* if for every compact set $E\subset\Omega$ there exists $C_E \geq 1$ so that

$$\|\rho\|_{n,B(a,r)} \leq C_E\|\rho\|_{n,B(a,r/2)} \qquad \text{(LD)}$$

whenever $B(a,r) \subset E$.

We also say that ρ is *uniformly asymptotically closed* if for every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\text{(UAC)} \qquad r\, \frac{\nparallel d\rho\|_{q,B(a,r)}}{\nparallel \rho\|_{n,B(a,r/2)}} < \varepsilon$$

whenever $B(a,r) \subset \Omega$ and $0 < r < \delta$.

We say that ρ is a *K-quasiconformal frame (in Ω)* if it satisfies the quasiconformality condition (QC), Condition (LD), and Condition (UAC). Furthermore, such ρ is a *strong K-quasiconformal frame (in Ω)* if it also satisfies (UAC) for some $q > n-1$. Notice that the differential of a quasiregular mapping defines a strong quasiconformal frame, see [13].

In what follows, we show that a strong quasiconformal frame in a domain is a strong quasiconformal frame at every point of the domain. We begin with the following weak reverse Hölder inequality for quasiconformal frames in a domain.

Theorem 9.1. *Suppose that ρ is a K-quasiconformal frame in a domain Ω. Then there exist $C \geq 1$ and $\eta > 0$, only depending on n, K and q, and $r_0 > 0$ so that*

$$\nparallel \rho\|_{n,B(a,r)} \leq C \nparallel \rho\|_{n-\eta,B(a,2r)}. \tag{9.1}$$

whenever $B(a,2r) \subset \Omega$ and $0 < r < r_0$.

Now, by a variant of Gehring's lemma [12, Corollary 14.3.1] and (9.1), we see that, under the assumptions of Theorem 9.1, we have that there exist $p > n$ and $C > 0$, only depending on n, K and q, so that

$$\nparallel \rho\|_{p,B(a,r)} \leq C \nparallel \rho\|_{n,B(a,2r)}$$

whenever $B(a,2r) \subset \Omega$. In particular we have the following corollary.

Corollary 9.2. *Suppose that ρ is a (strong) K-quasiconformal frame in Ω. Then ρ is a (strong) K-quasiconformal frame at every $x_0 \in \Omega$.*

We begin the proof of Theorem 9.1 by quoting a result from [11]. Let $B = B(x,r)$. The averaged Poincaré homotopy operator $\mathcal{T}\colon L^p(\bigwedge^\ell B) \to L^p(\bigwedge^\ell B)$, defined in [11], is a chain homotopy between identity and zero in $W_{p,p}$ for $1 < p < \infty$, that is,

$$\omega = \mathcal{T} d\omega + d\mathcal{T}\omega \tag{9.2}$$

for every $\omega \in W_{p,p}(\bigwedge^\ell B)$. Moreover, $\mathcal{T}$ supports a *Sobolev-Poincaré inequality*

$$\nparallel \omega - \omega_B\|_{np/(n-p),B} \leq Cr \nparallel d\omega\|_{p,B} \tag{9.3}$$

for every $1 < p < n$, where

$$\omega_B = \begin{cases} d\mathcal{T}\omega, & \ell \geq 1, \\ |B|^{-1} \int_B \omega, & \ell = 0, \end{cases}$$

see [11, Corollary 4.2] for details.

Proof of Theorem 9.1. Fix $\varepsilon > 0$ to be determined later. We fix a ball $B = B(x,r) \subset \Omega$ with $r < \delta$, where δ is as in (UAC). We define an $(n-1)$-form ω by

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} (\mathcal{T}\rho_i - (\mathcal{T}\rho_i)_B) \cdot \rho_1 \wedge \cdots \wedge \hat{\rho}_i \wedge \cdots \wedge \rho_n.$$

Then, by (9.2), we have

$$d\omega \quad = \quad n\rho_1 \wedge \cdots \wedge \rho_n - \lambda_1 + \lambda_2,$$

where

$$\lambda_1 = \sum_{i=1}^{n} (-1)^{i-1} \left(\mathcal{T} d\rho_i\right) \wedge \rho_1 \wedge \cdots \wedge \hat{\rho}_i \wedge \cdots \wedge \rho_n$$

and

$$\lambda_2 = \sum_{i=1}^{n} (-1)^{i-1} (\mathcal{T}\rho_i - (\mathcal{T}\rho_i)_B) \cdot d(\rho_1 \wedge \cdots \wedge \hat{\rho}_i \wedge \cdots \wedge \rho_n).$$

For λ_1, λ_2, and ω, we have the pointwise estimates

$$|\lambda_1| \le C |\mathcal{T} d\rho|\, |\rho|^{n-1}, \ |\lambda_2| \le C |\mathcal{T}\rho - (\mathcal{T}\rho)_B|\, |d\rho|\, |\rho|^{n-2},$$

and

$$|\omega| \le C |\mathcal{T}\rho - (\mathcal{T}\rho)_B|\, |\rho|^{n-1}$$

almost everywhere in B, where $C > 0$ depends only on n. Here

$$\mathcal{T}\rho - (\mathcal{T}\rho)_B = (\mathcal{T}\rho_1 - (\mathcal{T}\rho_1)_B, \ldots, \mathcal{T}\rho_n - (\mathcal{T}\rho_n)_B).$$

We fix a test function $\phi \in C_0^\infty(B(x,r))$ so that $0 \le \phi \le 1$, $\phi | B(x,r/2) = 1$ and $|\nabla \phi| \le 3/r$. Then, by Stokes' theorem,

$$C^{-1} \int_B \phi \star (\rho_1 \wedge \cdots \wedge \rho_n) \le \int_B \left(\phi\, |\lambda_1| + \phi\, |\lambda_2| + |\omega|/r \right),$$

which, in view of our quasiconformality condition and pointwise estimates, yields

(9.4)

$$\begin{aligned} C^{-1} \nparallel \rho\|^n_{n,B(x,r/2)} \le & \, r^{-n} \int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B|\, |d\rho|\, |\rho|^{n-2} \\ & + r^{-n-1} \int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B|\, |\rho|^{n-1} + \ r^{-n} \int_B |\mathcal{T} d\rho|\, |\rho|^{n-1}, \end{aligned}$$

where $C > 0$ depends only on n and K.

We estimate each term separately. In what follows, we denote by t' the Hölder conjugate exponent of $1 < t < \infty$. Also, we denote $\nparallel \cdot \|_p = \nparallel \cdot \|_{p,B}$. We may assume that $n/2 < q < n$.

Set $k = nq/(n-q)$. Since $q > n/2$, we have that $k > n$ and $k'(n-1) < n$. By Hölder's inequality and the Sobolev-Poincaré inequality (9.3),

$$\begin{aligned} r^{-n-1}\int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B||\rho|^{n-1} &\leq Cr^{-1} \nparallel \mathcal{T}\rho - (\mathcal{T}\rho)_B\|_k \nparallel \rho\|^{n-1}_{k'(n-1)} \\ &\leq C \nparallel \rho\|_q \nparallel \rho\|^{n-1}_{k'(n-1)}. \end{aligned}$$

Similarly,

$$r^{-n}\int_B |\mathcal{T}d\rho||\rho|^{n-1} \leq Cr \nparallel d\rho\|_q \nparallel \rho\|^{n-1}_{k'(n-1)}$$

by (9.2) and (9.3). By our assumption, $r \nparallel d\rho\|_q \leq \varepsilon \nparallel \rho\|_{n,B(x,r/2)}$, so we have

$$\begin{aligned} &r^{-n}\int_B |\mathcal{T}d\rho||\rho|^{n-1} + r^{-n-1}\int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B||\rho|^{n-1} \\ &\qquad \leq C \nparallel \rho\|^{n-1}_{k'(n-1)}\big(\nparallel \rho\|_q + \varepsilon \nparallel \rho\|_{n,B(x,r/2)}\big). \end{aligned} \tag{9.5}$$

Set next $\nu = (n/2+q)/2$. Since $n/2 < \nu < q$, we have that $\nu'(n-2) < n$, and Hölder's inequality yields

$$\begin{aligned} &r^{-n}\int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B|\,|d\rho|\,|\rho|^{n-2} \\ &\qquad \leq C\Big(r^{-n}\int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B|^{\nu}\,|d\rho|^{\nu}\Big)^{1/\nu} \nparallel \rho\|^{n-2}_{\nu'(n-2)}. \end{aligned}$$

Furthermore, by Hölder's inequality, (9.3), and (UAC), we have

$$\begin{aligned} \Big(r^{-n}\int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B|^{\nu}\,|d\rho|^{\nu}\Big)^{1/\nu} &\leq C \nparallel \mathcal{T}\rho - (\mathcal{T}\rho)_B\|_{\nu(q/\nu)'} \nparallel d\rho\|_q \\ &\leq Cr \nparallel \rho\|_{\alpha} \nparallel d\rho\|_q \\ &\leq C\varepsilon \nparallel \rho\|_{\alpha} \nparallel \rho\|_{n,B(x,r/2)}, \end{aligned}$$

where $1 \leq \alpha < n$ satisfies $n\alpha/(n-\alpha) = \max\{\nu(q/\nu)', n/(n-1)\}$. By choosing $\varepsilon > 0$ small enough, we have that

$$r^{-n}\int_B |\mathcal{T}\rho - (\mathcal{T}\rho)_B|\,|d\rho|\,|\rho|^{n-2} \leq \nparallel \rho\|_{\alpha} \nparallel \rho\|_{n,B(x,r/2)} \nparallel \rho\|^{n-2}_{\nu'(n-2)}. \tag{9.6}$$

We denote $t = \max\{k'(n-1), q, \nu'(n-2), \alpha\} < n$. We may assume that $\nparallel \rho\|_t \leq \nparallel \rho\|_{n,B(x,r/2)}$, otherwise (9.1) holds. Then, by (9.4), (9.5), and (9.6),

$$\begin{aligned} \nparallel \rho\|^n_{n,B(x,r/2)} &\leq C \nparallel \rho\|^{n-1}_t (\nparallel \rho\|_t + \nparallel \rho\|_{n,B(x,r/2)}) \\ &\leq C \nparallel \rho\|^{n-1}_t \nparallel \rho\|_{n,B(x,r/2)}, \end{aligned}$$

where $C > 0$ depends only on n, K and q. The proof is complete. □

By Theorem B and Corollary 9.2, we know that if ρ is a strong quasiconformal frame in Ω, then the local degree $i(x,\rho)$ is well-defined and positive at every point $x \in \Omega$. We define the *branch set* $\mathcal{B}_\rho$ of a strong quasiconformal frame ρ to be the set of points $x \in \Omega$ for which $i(x,\rho) \geq 2$.

In [16], [10], and [7], *Cartan-Whitney presentations*, i.e. $W_{\infty,\infty}$-frames satisfying

$$\star(\rho_1 \wedge \cdots \wedge \rho_n) \geq \delta > 0 \tag{9.7}$$

almost everywhere, are considered. It is shown that for these frames the map f_ρ, defined as in Section 3, is a quasiregular mapping (in fact even a mapping of bounded length distortion) in a neighborhood of every basepoint x_0, and that the differential of f_ρ is close to the frame in norm. This gives a version of the measurable Riemann mapping theorem in this setting: the Beltrami system induced by a Cartan-Whitney presentation always has local approximative solutions. This in turn has significant applications to smoothability and bi-Lipschitz parametrization problems. In these applications the branch set of the frame is important, since it represents an obstruction to parametrizations. In [10] and [7] it is shown that the branch set has measure zero and topological dimension at most $(n-2)$. Also, in [7] a sharp additional assumption has been found, so that (9.7) and this assumption together imply that the branch set is empty. See [6] for further discussion. In our current work we have studied extensions where the "branched bi-Lipschitz" assumption (9.7) is replaced by a quasiconformality assumption. Instead of local approximative solutions, we found weaker infinitesimal solutions. As in the case of Cartan-Whitney presentations, also in the current setting it would be interesting to know properties of the branch set and to find out natural additional assumptions that force the branch set to be empty.

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