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MAPPINGS OF FINITE DISTORTION AND WEIGHTED PARABOLICITY

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Abstract. Mappings of finite distortion between Riemannian manifolds having a weighted parabolic manifold as a domain are studied. We prove that local homeomorphisms of finite distortion are essentially m-to-1 mappings, where m depends only on the fundamental groups of the manifolds. Some characterizations of weighted parabolic manifolds are also discussed.

1. Introduction

It is well known that a quasiregular local homeomorphism from $\mathbb{R}^n$ into itself is a homeomorphism if $n \geq 3$, see [Z1]. This theorem is known as Zorich's theorem or as the Global Homeomorphism Theorem. In the setting of Riemannian manifolds V.A. Zorich has proved a generalization of this theorem known as geometric version of the Global Homeomorphism Theorem.

Theorem 1 ([Z2]). *Let M be a parabolic Riemannian manifold and N a simply connected Riemannian manifold, both manifolds of dimension $n \geq 3$, and let $f\colon M \to N$ be a quasiregular local homeomorphism. Then f is a homeomorphism onto its image, and the set $N \setminus fM$ has zero n-capacity.*

Recently P. Koskela and J. Onninen proved in [KO] that a version of Väisälä's inequality for path families holds also for mappings of finite distortion satisfying some regularity assumptions. In [HP] I. Holopainen and the author proved a version of the Global Homeomorphism Theorem for mappings of finite distortion.

Theorem 2 ([HP]). *Let M be a $K^{n-1}(\cdot)$-parabolic Riemannian manifold, N a simply connected Riemannian manifold, both of dimension $n \geq 3$, and let $f\colon M \to N$ be a local homeomorphism of finite distortion $K(\cdot)$ satisfying the condition (A). Then f is a homeomorphism onto its image and the set $N \setminus fM$ has zero n-capacity.*

Mappings of finite distortion and the condition (A) are defined in the following section. The method of the proof allows us to consider the situation without the simply connectedness of N. Indeed, mappings of finite distortion from a parabolic manifold into another n-manifold are essentially m-to-1 mappings, where m depends only on the fundamental groups of M and N.

Theorem 3. *Let M be a $K^{n-1}(\cdot)$-parabolic Riemannian manifold, N a Riemannian manifold, both of dimension $n \geq 3$, and let $f\colon M \to N$ be a local homeomorphism of*

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finite distortion $K(\cdot)$ satisfying the condition (A). Then there exists a set $E \subset N$ of zero n-capacity such that f is an m-to-1 mapping on $M \setminus f^{-1}E$, where

$$m = \text{card } (\pi_1(N)/f_*\pi(M)) \in \mathbb{Z}_+ \cup \{\infty\},$$

and card $f^{-1}(y) < m$ *for every $y \in E$, if $m < \infty$. Moreover, $N \setminus fM \subset E$ and E is the set of asymptotic limits of f.*

In order to describe the idea of the proof let us define the weighted parabolicity of a Riemannian manifold. The other definitions are given in the following section. We say that a C^∞, oriented, connected Riemannian n-manifold M is (p,w)-parabolic, where $w\colon M \to [0,\infty]$ is a measurable function, if the path family

$$\Gamma_M^\infty = \{\gamma\colon |\gamma| \not\subset C,\ C \subset M \text{ compact}\}$$

has zero p-modulus with weight w, that is, $\mathsf{Mod}_{p,w}(\Gamma_M^\infty) = 0$. The path family Γ_M^∞, consisting of paths leaving every compact set, is called the family of paths going to the infinity. We say that M is w-parabolic, if $p = n$. It is equivalent to define (p,w)-parabolic manifolds by requiring that every compact subset C of M has zero w-weighted p-capacity with respect to M, that is, $\mathsf{cap}_{p,w}(C,M) = 0$. The weighted modulus and capacity are discussed in [KO]. The definitions are also given in the following section.

The proofs of Theorems 2 and 3 are based on the following ideas. First, by the parabolicity of M the set of asymptotic limits of f is very small, that is, having zero n-capacity. Secondly, paths outside the set of asymptotic limits admit total lifts. Hence we can also lift homotopies. Thirdly, the set of asymptotic limits is so small that removing it does not affect the fundamental group.

It should be noted that if the mapping f is not a local homeomorphism, then the set of asymptotic values can be large. See e.g. [D] where a quasiregular mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ having every point of $\mathbb{R}^n$ as an asymptotic limit is constructed.

This note is organized as follows. In Section 3 some characterizations of weighted parabolic manifolds are discussed. The sets of the asymptotic limits of mappings of finite distortion under the assumptions of Theorem 3 are considered in Section 4. Finally, Section 5 is devoted to the proof of Theorem 3.

2. Some definitions and preliminaries

In this note we consider only C^∞, oriented, connected, Riemannian n-manifolds without boundary. We always assume that $n \geq 3$. Let M and N be such manifolds.

We say that a continuous mapping $f\colon M \to N$ has *finite distortion*, if the mapping is locally in the class $W^{1,1}_{\text{loc}}(M,N)$, that is, the local representations of f are in the class $W^{1,1}_{\text{loc}}$, and f satisfies the following conditions:

(a) the Jacobian determinant $J(\cdot,f)$ of f is locally integrable, and
(b) there exists a measurable function $K\colon M \to [1,\infty]$, finite almost everywhere, such that
$$||Df(x)||^n \leq K(x)J(x,f) \quad \text{a.e. } x \in M.$$

Moreover, we assume that the function $K(\cdot)$ satisfies condition

$$\text{(A)} \qquad \exp\bigl(\mathcal{A}(K)\bigr) \in L^1_{\text{loc}}(M),$$

where $\mathcal{A}$ is an Orlicz function such that

$$\int_1^\infty \frac{\mathcal{A}'(t)}{t}\,\mathrm{d}t = \infty, \tag{A-1}$$

and

$$t\mathcal{A}'(t) \text{ increases to } \infty \text{ for sufficiently large } t. \tag{A-2}$$

The minimal integrability assumptions on the distortion function K are discussed in great detail in [IKO], [KKM], and [KKMOZ]. See also [HP, Sec. 2] for a discussion of the definition of mappings of finite distortion between Riemannian manifolds.

The w-weighted p-modulus of path family Γ in M is

$$\mathsf{Mod}_{p,w}(\Gamma) = \inf_\rho \int_M \rho^p w\,\mathrm{d}m, \tag{1}$$

where the infimum is taken over non-negative Borel functions ρ such that

$$\int_\gamma \rho\,\mathrm{d}s \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. The measure m in the definition of the weighted modulus is the measure given by the Riemannian metric of M. The w-weighted p-capacity of the condenser (C, Ω) in M is

$$\mathsf{cap}_{p,w}(C,\Omega) = \inf_u \int_M ||\nabla u||^p w\,\mathrm{d}m,$$

where the infimum is taken over functions $u \in C_0^\infty(\Omega)$ satisfying $u|C \geq 1$.

For a constant weight w it is well know that

$$\mathsf{cap}_{p,w}(C,\Omega) = \mathsf{Mod}_{p,w}(\Gamma)$$

when Γ is the family of all paths $\gamma \in \Gamma_\Omega^\infty$ intersecting the set C, see e.g. [R, Prop. II.10.2]. The proof given in [R, Prop. II.10.2] works almost verbatim also in the weighted case.

In the paper [KO] P. Koskela and J. Onninen show that Väisälä's inequality for modulus of path families holds for mappings of finite distortion when the distortion function $K(\cdot)$ satisfies the condition (A). As a corollary of this version of Väisälä's inequality we obtain a version of Poletsky's inequality which is sufficient for our considerations. Let $f\colon M \to N$ be a mapping of finite distortion $K(\cdot)$ satisfying the condition (A) and let Γ be a path family in M. Then

$$\mathsf{Mod}_n(f\Gamma) \leq \mathsf{Mod}_{n,K^{n-1}}(\Gamma).$$

See [KO] for a discussion on the modulus and the capacity inequalities for mappings of finite distortion in the Euclidean setting. The required modifications for Väisälä's inequality for mappings of finite distortion between Riemannian manifolds are discussed in [HP, Sec. 3].

We use following notations for the special type of path families. For every $E \subset M$ we denote by Γ_E the path family of all paths intersecting the set E. For every pair E and F of subsets of M we denote by $\Delta(E,F)$ the family of the paths connecting E and F. Moreover, we denote by $|\gamma|$ the locus of the path γ.

3. Characterization of weighted parabolic manifolds

In the unweighted case ($w \equiv 1$) it is known that a sufficient condition for p-parabolicity, $1 < p < \infty$, of a complete non-compact manifold is an Ahlfors type criterion

$$\int_1^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} \mathrm{d}t = \infty. \tag{2}$$

Here $V(t)$ is the volume of a ball of radius t around some fixed point. For details see e.g. [HK, 2.3]. In [HK] it is actually shown that the condition (2) is sufficient in the class of proper metric spaces equipped with Borel regular measure assigning every ball a finite positive measure. Thus, if we assume that $0 < ||w\chi_B||_1 < \infty$ for every ball B in M, we obtain that the condition (2) is sufficient for (p,w)-parabolicity of a complete Riemannian manifold M with the measure $\mathrm{d}\mu = w\ \mathrm{d}m$. The condition (2) is not necessary in general, see e.g. [H]. It is shown in [HK] that if the measure is doubling and supports a $(1,p)$-Poincaré inequality the condition (2) becomes necessary for p-parabolicity.

In the conformal case $p = n$ the parabolicity can be characterized using only completeness and volume. For locally integrable weights the characterization of weighted parabolicity is analogous to the unweighted case. Since the proof of the weighted case follows closely the proof of the unweighted case given in [ZK], we give only a sketch of the proof of the weighted case.

Theorem 4. *Let (M,g) be a n-Riemannian manifold and $w\colon M \to [0,\infty]$ be in $L^1_{\mathrm{loc}}(M)$. Then (M,g) is w-parabolic if and only if there exists C^∞ function $\lambda\colon M \to]0,\infty[$ such that $(M,\lambda g)$ is complete and*

$$||w||_{L^1(M,\lambda g)} = \int_M w\lambda^n\ \mathrm{d}m_g < \infty. \tag{3}$$

Here m_g is the measure given by the Riemannian metric g.

Sketch of the proof. We may assume that M is a non-compact complete manifold. Indeed, the claim is trivial for compact manifolds, and for a non-compact manifold M we may choose a function λ_0 such that $(M,\lambda_0 g)$ is complete. The metric $\lambda_0 g$ is conformally equivalent to g and hence this has no effect to the w-parabolicity.

Let us first consider the sufficiency. Let $\tilde{g} = \lambda g$. Since the n-modulus is a conformal invariant, it is sufficient to show that the (n,w)-modulus of Γ_M^∞ with respect to Riemannian metric $\tilde{g}$ is zero. Because paths in Γ_M^∞ have infinite length in the metric $\tilde{g}$, every constant positive function is admissible and thus, by the condition (3), we have that the family Γ_M^∞ has zero modulus.

Let us now consider the necessity. Let us fix a point $o \in M$. Since M is a complete w-parabolic manifold, we may choose an increasing sequence of radii $r_i \in]0,\infty[$ such that $r_i \to \infty$ as $i \to \infty$ and

$$\mathsf{cap}_{n,w}(\bar{B}(o,r_i), B(o,r_{i+1})) = \mathsf{Mod}_{n,w}(\Delta(\partial B(o,r_i), \partial B(o,r_{i+1})) < 2^{-i}.$$

Thus, by the definition of the capacity, there exist functions $u_i \in C_0^\infty(M)$ such that $u_i|\bar{B}(o,r_i) \geq 1$, $\operatorname{supp} u_i \subset B(o,r_{i+1})$, and

$$\int_M ||\nabla u_i||^n w\ \mathrm{d}m_g \leq 2^{-i}.$$

Moreover, since $w \in L^1_{\mathrm{loc}}(M)$, we may choose a positive C^∞ function $\varphi\colon M \to]0,\infty[$ such that
$$\int_{\bar{B}(o,r_{i+1})\setminus B(o,r_i)} \varphi w \, \mathrm{d}m_g \le 2^{-i}.$$
Now it is easy to show that the function
$$\lambda = \left(\varphi + \sum_{i=1}^{\infty} ||\nabla u_i||^n\right)^{1/n}$$
satisfies the claimed properties. □

4. The set of asymptotic limits

In this section we show that under the assumptions of Theorem 3 the set of asymptotic limits of f has a zero n-capacity. In particular our objective is to discuss the role of the local homeomorphicity of f and the dimension assumption $n \ge 3$.

Let $f\colon M \to N$ be a mapping between manifolds M and N. We say that a point $y \in N$ is an asymptotic limit of f, if there exists a path $\gamma\colon [a,b[\to M$ such that $\gamma \in \Gamma_M^\infty$ and $f \circ \gamma(t) \to y$ as $t \to b$. The set of all asymptotic limits of f is denoted by $E(f)$.

It is shown in [Z2] that the set the asymptotic limits of a quasiregular local homeomorphism from an n-parabolic manifold into another n-manifold has zero n-capacity. The same method can be applied to local homeomorphisms of finite distortion, see [HP, Sec. 4] for details. In this section we give an outline of the proof given in [HP, Sec. 4].

Proposition 5 ([HP, Prop. 2]). *Let M and N be connected oriented Riemannian manifolds of dimension $n \ge 3$ and let $f\colon M \to N$ be a local homeomorphism of finite distortion $K(\cdot)$ satisfying condition (A). If M is $K(\cdot)^{n-1}$-parabolic, then $\mathsf{cap}_n(E(f)) = 0$ and $E(f)$ is σ-compact. Moreover, $N \setminus fM \subset E(f)$.*

Let us now discuss the idea of the proof of Proposition 5. By the connection between capacity and modulus, it is sufficient to show that $\mathsf{Mod}_n(\Gamma_{E(f)}) = 0$. Hence, it suffices to show that $\Gamma_{E(f)} < f\Gamma_M^\infty$, that is, every path γ in N intersecting the set $E(f)$ has a subpath γ', which has a lift σ in f such that $\sigma \in \Gamma_M^\infty$. Indeed, if $\Gamma_{E(f)} < f\Gamma_M^\infty$ holds we have by Poletsky's inequality and the $K^{n-1}(\cdot)$-parabolicity of M that
$$\mathsf{Mod}_n(\Gamma_{E(f)}) \le \mathsf{Mod}_n(f\Gamma_M^\infty) \le \mathsf{Mod}_{n,K^{n-1}}(\Gamma_M^\infty) = 0.$$
However, by the definition of the set $E(f)$, we know only that for every point in $E(f)$ there exists some path which has a lift in f belonging to Γ_M^∞.

Since the capacity and the σ-compactness are preserved in countable unions, it is sufficient to consider a countable set of bilipschitz charts (U,φ) satisfying conditions $\bar{B}^n \subset \varphi U$ and $\varphi^{-1}(0) \in N \setminus E(f)$, and to show that for every such chart (U,φ) the set $E(f) \cap U$ is σ-compact and has zero n-capacity. See [HP, Prop. 2] for details how these charts can be chosen. Let $E(f)_U = \varphi(E(f) \cap U)$ for every chosen chart (U,φ).

The properties of the paths visiting the set $E(f) \cap U$ are now studied in three steps. First, we find an maximal open star-shaped set $G \subset B^n$ around origin such that for every $z \in G$ and every path from origin to z in G has a total lift in $\varphi \circ f|f^{-1}U$ starting from x_U. Hence $E(f)_U \subset B^n \setminus G$. The second step is to show that the set G is relatively locally connected, G is dense in B^n, and $B^n \setminus G$ has zero n-dimensional measure. The modulus estimate given in [Z2, Lemma 2] is crucial for this step. See also [HP, Lemma 12, Lemma 25] for details. In the third step, we show that every path in $\Gamma_{E(f)\cap U}$ has a subpath in $f\Gamma_M^\infty$. Indeed, for every point $y \in E(f) \cap U$ we find a path $\sigma\colon [0,\infty[\to M$ such that $\sigma \in \Gamma_M^\infty$ and $\varphi \circ f \circ \sigma(t) \to \varphi(y)$ as $t \to \infty$. Now for every path γ in B^n we may use the relative local connectedness of G, the local homeomorphicity of f, and the path $\varphi \circ f \circ \sigma$ to construct a path $\sigma' \in \Gamma_M^\infty$ such that $\varphi \circ f \circ \sigma'$ is a subpath of γ.

It should be noted that the dimension assumption, $n \geq 3$, was used only to obtain the relative local connectedness of the set G. In dimension 2 this property does not hold. Indeed, for $n = 2$ every ray in B^2 separates it locally. Simple examples, say the exponential mapping, show that all versions of the Global Homeomorphism Theorem fail when $n = 2$. The local homeomorphicity of f was used throughout the proof, in particular in path-lifting.

5. Proof of Theorem 3

For the proof of Theorem 3 we need the following proposition.

Proposition 6 ([HP, Prop. 3]). *Let N be a connected n-manifold, and $E \subset N$ a σ-compact set such that $\mathcal{H}^{n-2}(E) = 0$. Then groups $\pi_1(N \setminus E)$ and $\pi_1(N)$ are isomorphic, and $N \setminus E$ is path-connected.*

A similar proposition is originally proved in [MRV] for compact sets. Since we need the result for the set of asymptotic limits, which we know only to be σ-compact, we need a refined version of the original result. See [HP] for the details of the proof. The author does not know whether the σ-compactness of E can be relaxed. For the proof of Theorem 3 some properties of the set $M \setminus f^{-1}E(f)$ are still to be verified. For quasiregular local homeomorphisms it is well known that the inverse image of a set of zero n-capacity has also zero n-capacity. However, the following lemma is sufficient for our considerations.

Lemma 7. *Let manifolds M and N and the mapping f be as in the Theorem 3. For every path $\gamma\colon [0,1] \to M$ such that $\gamma(0)$ and $\gamma(1)$ belong to $M \setminus f^{-1}E(f)$ there exists a path $\sigma\colon [0,1] \to M \setminus f^{-1}E(f)$ homotopic to γ and having the same endpoints as γ.*

Proof. Let $\gamma\colon [0,1] \to M$ be a path such that $\gamma(0)$ and $\gamma(1)$ belong to $M \setminus f^{-1}E(f)$. Let $\{U_i\}$, $i = 0,\ldots,k$, be a finite cover of $|\gamma|$ such that $U_i \cap U_{i+1} \neq \emptyset$, $\gamma(0) \in U_0$, $\gamma(1) \in U_k$, and $f|U_i\colon U_i \to fU_i$ is a homeomorphism. Moreover, we may assume that fU_i is homeomorphic to B^n for every i. Let $0 = t_0 < t_1 < \ldots < t_k = 1$ be a partition of $[0,1]$ such that $\gamma[t_i,t_{i+1}] \subset U_i$ for every i. Since $E(f)$ has zero n-capacity, we may assume that $f(\gamma(t_i)) \notin E(f)$ for every i. For every $i = 0,\ldots,k-1$ there exists a path $\sigma_i\colon [t_i,t_{i+1}] \to fU_i \setminus E(f)$ such that $\sigma_i(t_i) = f(\gamma(t_i))$ and $\sigma_i(t_i) = f(\gamma(t_{i+1}))$. Indeed, we may consider the set fU_i as a manifold and apply Proposition 6 to the set $fU_i \cap E(f)$. Since every path σ_i is homotopic to $f \circ \gamma|[t_i,t_{i+1}]$, the path $\sigma\colon [0,1] \to M \setminus f^{-1}E(f)$, $\sigma|[t_i,t_{i+1}] = (f|U_i)^{-1} \circ \sigma_i$, is homotopic to γ. □

In the proof of Theorem 3 we assume that every loop is defined on the unit interval $[0,1]$. Moreover, for loops α and β starting at the same point, we denote by $\alpha * \beta$ the loop which is the concatenation of these loops. The inverse of a path γ is denoted by γ^{-1}.

Proof of Theorem 3. By Proposition 5 the set $E(f)$ has zero n-capacity. We show that it satisfies also the other properties.

Let us fix an arbitrary point $y \in N \setminus E(f)$ and choose $x \in f^{-1}(y)$. For every class $c \in \pi_1(N)/f_*\pi_1(M)$ let us fix a point $x_c \in f^{-1}(y)$ in the following way. For $c \in \pi_1(N)/f_*\pi_1(M)$ choose a loop α_c starting from y. By Proposition 6, we may choose α_c in such way that $|\alpha_c| \subset N \setminus E(f)$. Hence, it has a total lift β_c starting from x. Thus we may choose $x_c = \beta_c(1)$. Let c and c' be classes in $\pi_1(N)/f_*\pi_1(M)$ and suppose $x_c = x_{c'}$. Then the paths β_c and $\beta_{c'}$ have the same end points and thus the path $\beta_c * (\beta_{c'})^{-1}$ is a loop in M starting at x. Hence the homotopy classes of paths α_c and $\alpha_{c'}$ belong to the same class in $\pi_1(N)/f_*\pi_1(M)$. Thus $c = c'$. We conclude that

$$\operatorname{card} f^{-1}(y) \geq \operatorname{card} \left(\pi_1(N)/f_*\pi_1(M)\right).$$

Let x_0 and x_1 be points in $f^{-1}(y)$ and let β_0 and β_1 be paths joining x to x_0 and x to x_1 in $M \setminus f^{-1}E(f)$, respectively. Then the homotopy classes of paths $f \circ \beta_0$ and $f \circ \beta_1$ represent some classes c_0 and c_1 in $\pi_1(N)/f_*\pi_1(M)$. If $c_0 = c_1$, there exists a loop γ in M starting at x such that the paths $f \circ \beta_0$ and $(f \circ \gamma) * (f \circ \beta_1)$ are homotopic in $N \setminus E(f)$. We may, by Lemma 7, assume that $|\gamma| \subset M \setminus f^{-1}E(f)$. Moreover, we have that $f \circ (\gamma * \beta_1) = (f \circ \gamma) * (f \circ \beta_1)$. Let us now fix a homotopy $F\colon [0,1] \times [0,1] \to N \setminus E(f)$ between the paths $f \circ \beta_0$ and $f \circ (\gamma * \beta_1)$. There exists a homotopy $H\colon [0,1] \times [0,1] \to M$ starting from β_0 such that $f \circ H = F$. By the uniqueness of path lifting, we conclude that H is a homotopy between β_0 and $\gamma * \beta_1$. Thus the end points x_0 and x_1 of paths β_0 and β_1, respectively, are the same. This yields the inequality

$$\operatorname{card} f^{-1}(y) \leq \operatorname{card} \pi_1(N)/f_*\pi_1(M).$$

Let us now fix a point $y \in E(f) \cap fM$. We may assume that $\operatorname{card} \pi_1(N)/f_*\pi_1(M) < \infty$, and hence $\operatorname{card} f^{-1}(z) < \infty$ for every $z \in N \setminus E(f)$. Suppose $\operatorname{card} f^{-1}(y) \geq m$. Then we may choose m points $x_i \in f^{-1}(y)$ and their disjoint relatively compact neighborhoods U_i such that $f|U_i\colon U_i \to fU_i$ is a homeomorphism. Let $\alpha\colon [a,b] \to N$ be a path such that $\alpha(b) = y$. Let us fix some $z \in |\alpha| \cap fU_1 \cap \cdots \cap fU_m \setminus E(f)$. Since $\operatorname{card} f^{-1}(z) = m$, we have that $f^{-1}(z) \subset U_1 \cup \cdots \cup U_m$. Thus no lift of α is in Γ_M^∞. Since α was arbitrary, this contradicts the assumption $y \in E(f)$. Therefore $\operatorname{card} f^{-1}(y) < m$. □

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